

Uncertainty and the Social Planner’s Problem: Why Sample Complexity Matters

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Welfare measures overall utility across a population, whereas malfare measures overall disutility, and the social planner’s problem can be cast either as maximizing the former or minimizing the latter. We show novel bounds on the expectations and tail probabilities of estimators of welfare, malfare, and regret of per-group (dis)utility values, where estimates are made from a finite sample drawn from each group. In particular, we consider *estimating* these quantities for individual functions (e.g., allocations or classifiers) with standard probabilistic bounds, and optimizing and *bounding generalization error over hypothesis classes* (i.e., we quantify overfitting) using Rademacher averages. We then study algorithmic fairness through the lens of sample complexity, finding that because marginalized or minority groups are often understudied, and fewer data are therefore available, the social planner is more likely to overfit to these groups, thus even models that *seem fair in training* can be *systematically biased* against such groups. We argue that this effect can be mitigated by ensuring sufficient sample sizes for each group, and our sample complexity analysis characterizes these sample sizes. Motivated by these conclusions, we present *progressive sampling* algorithms to efficiently optimize various fairness objectives.

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1 INTRODUCTION

Machine learning systems in settings like facial recognition [10, 11, 13] and medicine [3, 4, 12] exhibit *differential accuracy* across race and other protected groups. This can lead to discrimination: for example, facial recognition in policing yields disproportionate false-arrest rates [24], and machine learning in medicine can lead to inequity of health outcomes [18], both of which exacerbate existing structural inequalities impacting minority groups. In recent years, researchers have proposed welfare-centric fair learning models, which constrain or optimize welfare [15, 22, 26, 28, 38, 44, 45] or malfare [1, 14, 20, 30, 33, 43] to promote *fair learning* across *all groups*, as well as regret-based methods [7, 39], which similarly promote fairness by minimizing the *maximum dissatisfaction* of any group, relative to their preferred outcome, i.e., how much excess risk is incurred, or utility is lost, to any group by *compromising* on a *shared model*.

We study sampling and learning problems in the optimization of welfare, malfare, and regret objectives. In particular, our setting subsumes the *minimax fair learning* [1, 20, 30, 33, 43] and *fair-PAC learning* [14] settings, by optimizing arbitrary malfare or welfare functions, as well as the *multi-group agnostic PAC learning* [7, 39] setting, by considering *arbitrary malfare functions* (rather than just the maximum) of per-group regret values. This extension naturally and smoothly interpolates between minimizing *utilitarian* (i.e., weighted average) and *egalitarian* (i.e., maximum) malfare of risk or regret. Crucially, this allows for fine-grained control over the desired fairness concept, and mitigates the minority rule issues of minimax methods, while remaining axiomatically grounded in *cardinal welfare theory*. We bound the generalization error of optimizing welfare, malfare, and regret objectives, and find that while the power-mean

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malfare is always easy to estimate, due to Lipschitz-continuity (as studied by Cousins [14]), our learning algorithms work for any malfare, welfare, or regret objective that is *continuous* and *monotonic* in per-group (dis)utility values.

We then study algorithmic fairness through the lens of sample complexity, finding that because marginalized or minority groups are often understudied, and fewer data are therefore available, the social planner is more likely to *overfit* to these groups. Consequently, even models that *seem fair* in training can be *systematically biased* against such groups. Section 3 shows that this effect can be mitigated with sufficient per-group sample sizes, and §4 presents *progressive sampling* methods, which dynamically sample until a near-optimal model (w.r.t. some fairness objective) is learned.

Our bounds leverage the specific character of the objective at hand; for example, utilitarian welfare is sensitive to the *average* confidence radius across groups, whereas egalitarian welfare is more sensitive to the confidence radii of disadvantaged (i.e., low-utility or high-risk) groups. Furthermore, our progressive sampling methods are tailored to three realistic models of data generation: in the *joint sampling* model, each sample contains a piece of information for every group, in the *mixture sampling* model, samples are annotated with sets of group labels, and in the *conditional sampling* model, we are allowed to choose from which groups to sample. While our settings and modeling assumptions are practically motivated, this is a highly theoretical paper, and all novel results are meticulously proven in appx. A.

2 LEARNING FRAMEWORK AND OBJECTIVES

In this section, we introduce the functional forms of the objects and random spaces that we operate over, and we define our learning objectives. In particular, §2.1 presents the welfare, malfare, and regret objectives, which compile per-group sentiment values into a *cardinal objective value* that can be optimized and analyzed, then §2.2 reifies this abstract mathematics with three realistic models of data-collection, each of which requires its own statistical treatment to efficiently learn from data, i.e., to optimize and bound objectives, while minimizing the cost of obtaining said data.

We henceforth assume a *supervised learning setting*, where \mathcal{X} is the domain and \mathcal{Y} is the codomain. We also assume either a *loss function*¹ $\ell(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$ or a *utility function* $u(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$, which map predictions and labels onto *negatively connoted* loss or disutility, or *positively connoted* gain or utility, generically termed a *sentiment function* $s(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$. In most supervised learning settings, a single probability distribution \mathcal{D} over $\mathcal{X} \times \mathcal{Y}$ suffices, but we assume a set \mathcal{Z} of g groups, and we model the experiences and conditions of each group as its own distribution, i.e., we have $\mathcal{D}_1, \dots, \mathcal{D}_g$. For convenience, we often compose the sentiment function with a predictor or model $h(\cdot) : \mathcal{X} \rightarrow \mathcal{Y}'$, taking $(s \circ h)(x, y) \doteq s(h(x), y)$, thus we quantify model performance for group i as $\mathbb{E}_{\mathcal{D}_i} [s \circ h]$.

2.1 Fair Learning with Malfare, Welfare, and Regret Objectives

Here we define the welfare, malfare, and regret objectives. While the details differ, each of these is a function of the expected utility or loss (generically sentiment) of some $h : \mathcal{X} \rightarrow \mathcal{Y}'$ for each of the g groups, and we are interested in selecting the model or hypothesis h from some *hypothesis class* $\mathcal{H} \subseteq \mathcal{X} \rightarrow \mathcal{Y}'$ that optimizes the given objective.

Malfare and Welfare. A welfare function $W(\mathbf{s}; \mathbf{w})$ measures overall positive utility \mathbf{s} across a population weighted by \mathbf{w} , whereas a malfare function $M(\mathbf{s}; \mathbf{w})$ measures overall disutility \mathbf{s} , and generically, we say an aggregator function $M(\mathbf{s}; \mathbf{w})$ measures overall sentiment \mathbf{s} . The prototypical example is the *utilitarian* (or *Benthamite*) aggregate, defined as $M_1(\mathbf{s}; \mathbf{w}) \doteq \mathbf{s} \cdot \mathbf{w}$, which simply averages sentiment across the population (e.g., welfare as per-capita income, or malfare as per-capita medical expenditure), and the second-fiddle is the *egalitarian* (or *Rawlsian*) welfare $W_{-\infty}(\mathbf{s}; \mathbf{w})$ (minimum)

¹Often $\mathcal{Y}' = \mathcal{Y}$, such as in standard classification and regression settings, but this is not universally the case. For instance, in probabilistic classification or regression (i.e., conditional density estimation), \mathcal{Y}' is a space of *distributions over* \mathcal{Y} , and in *interval estimation*, \mathcal{Y}' is a space of *sets over* \mathcal{Y} .

or malfare $\Lambda_\infty(\mathbf{s}; \mathbf{w})$ (maximum), which summarizes a population's sentiment as that of its most disadvantaged member. We assume throughout that $\mathbf{w} \in (0, 1)^g$ is a *probability vector*, thus $\|\mathbf{w}\|_1 = 1$, and $\mathbf{s} \in \mathbb{R}_{0+}^g$ is *nonnegative*. *Ab initio*, our first objective is, via the social planner's problem, to maximize welfare [15, 22, 28, 38, 44], or by extension (e.g., in chores manna or harm allocation [25, 29], or in machine learning [1, 14]) to minimize malfare, i.e., we seek to approximate

$$h^* \doteq \operatorname{argmin}_{h \in \mathcal{H}} \Lambda \left(i \mapsto \mathbb{E}_{\mathcal{D}_i} [t \circ h]; \mathbf{w} \right), \quad \text{or} \quad h^* \doteq \operatorname{argmax}_{h \in \mathcal{H}} W \left(i \mapsto \mathbb{E}_{\mathcal{D}_i} [u \circ h]; \mathbf{w} \right). \quad (1)$$

Intuitively, the utilitarian case seeks to optimize overall or average sentiment, whereas the egalitarian case instead seeks to lift up the most disadvantaged group, and thus promote equality, perhaps at the expense of overall (total) utility.

Notably, welfare maximization generalizes *utility maximization* to multiple groups, and malfare minimization likewise generalizes *risk minimization*, and the well-studied *minimax fair learning* framework arises as the special-case of *egalitarian malfare minimization*. In general, we assume only *monotonicity* and *continuity* of aggregator functions; however, there are a set of relatively standard axioms that, when taken together, restricts the class of interest to the *power-mean family* [14]. This is convenient, as all power-mean malfare functions are Lipschitz-continuous, which in §4 leads to stronger estimation guarantees and more efficient sampling algorithms than ε - δ limit-continuity.

Definition 2.1 (Axioms of Cardinal Welfare and Malfare). Suppose an aggregator function $M(\mathbf{s}; \mathbf{w})$. For each item, assume (if necessary) that the axiom applies for all $\mathbf{s}, \mathbf{s}' \in \mathbb{R}_{0+}^g$, scalars $\alpha, \beta \in \mathbb{R}_{0+}$, and probability vectors $\mathbf{w} \in (0, 1)^g$.

- 1) *Strict Monotonicity*: $\mathbf{s}' \neq \mathbf{0} \implies M(\mathbf{s}; \mathbf{w}) < M(\mathbf{s} + \mathbf{s}'; \mathbf{w})$.
- 2) *Weighted Symmetry*: Suppose $g' \in \mathbb{Z}_+$, $\mathbf{s}' \in \mathbb{R}_{0+}^{g'}$, and probability vector $\mathbf{w}' \in (0, 1)^{g'}$, such that for all $u \in \mathbb{R}_{0+}$, it holds that $\sum_i \text{s.t. } s_i = u \mathbf{w}_i = \sum_i \text{s.t. } s'_i = u \mathbf{w}'_i$. Then $M(\mathbf{s}; \mathbf{w}) = M(\mathbf{s}'; \mathbf{w}')$.
- 3) *Continuity*: $M(\mathbf{s}; \mathbf{w})$ is a continuous function (in the standard ε - δ limit-continuity sense) in both \mathbf{s} and \mathbf{w} .
- 4) *Independence of Unconcerned Agents*: $M(\langle s_{1:g-1}, \alpha \rangle; \mathbf{w}) \leq M(\langle s'_{1:g-1}, \alpha \rangle; \mathbf{w}) \implies M(\langle s_{1:g-1}, \beta \rangle; \mathbf{w}) \leq M(\langle s'_{1:g-1}, \beta \rangle; \mathbf{w})$.
- 5) *Multiplicative Linearity*: $M(\alpha \mathbf{s}; \mathbf{w}) = \alpha M(\mathbf{s}; \mathbf{w})$.
- 6) *Unit Scale*: $M(\mathbf{1}; \mathbf{w}) = M(\langle 1, \dots, 1 \rangle; \mathbf{w}) = 1$.
- 7) *Pigou-Dalton Transfer Principle*: Suppose $\mu = \mathbf{w} \cdot \mathbf{s} = \mathbf{w} \cdot \mathbf{s}'$, and for all $i \in \mathcal{Z}$: $|\mu - s'_i| \leq |\mu - s_i|$. Then for utility and welfare, $W(\mathbf{s}'; \mathbf{w}) \geq W(\mathbf{s}; \mathbf{w})$, and for disutility and malfare, $\Lambda(\mathbf{s}'; \mathbf{w}) \leq \Lambda(\mathbf{s}; \mathbf{w})$.

Axioms 1–4 are essentially the standard *axioms of cardinal welfare* [37, 40], modified to include the weights \mathbf{w} , and omitting any of them leads to rather perverse aggregator functions. Axiom 5 (multiplicative linearity) strengthens the traditional *independence of common scale* axiom, and ensures that the *units* of welfare or malfare must match those of sentiment, and axiom 6 (unit scale) merely specifies a multiplicative constant. Finally, axiom 7, the *Pigou-Dalton transfer principle* [17, 36], characterizes *fairness* in the sense of *equitable redistribution* of utility (welfare) or disutility (malfare).

Theorem 2.2 (Aggregator Function Properties [14, thms. 2.4 and 2.5]). Suppose aggregator function $M(\mathbf{s}; \mathbf{w})$, and assume arbitrary sentiment vector $\mathbf{s} \in \mathbb{R}_{0+}^g$ and probability vector $\mathbf{w} \in (0, 1)^g$. The following then hold.

- 1) *Power-Mean Factorization*: Axioms 1–6 imply $\exists p \in \mathbb{R}$ s.t.

$$M(\mathbf{s}; \mathbf{w}) = M_p(\mathbf{s}; \mathbf{w}) \doteq f_p^{-1} \left(\sum_{i \in \mathcal{Z}} \mathbf{w}_i f_p(s_i) \right) =_{p \neq 0} \sqrt[p]{\sum_{i=1}^g \mathbf{w}_i s_i^p}, \quad \text{with} \quad \begin{cases} p = 0 & f_0(x) \doteq \ln(x) \\ p \neq 0 & f_p(x) \doteq \operatorname{sgn}(p) x^p \end{cases}.$$

- 2) *Fair Welfare and Malfare*: Axioms 1–7 imply $p \in (-\infty, 1]$ for welfare and $p \in [1, \infty)$ for malfare.

- 3) *Lipschitz-Continuity*: For all $p \geq 1$, it holds that $|M_p(\mathbf{s}; \mathbf{w}) - M_p(\mathbf{s}'; \mathbf{w})| \leq M_p(|\mathbf{s} - \mathbf{s}'|; \mathbf{w}) \leq \|\mathbf{s} - \mathbf{s}'\|_\infty$.

In closing, we note that utilitarian philosophy is often criticized for permitting great inequality by ignoring the needs of smaller or less visible groups, whereas egalitarian philosophy is criticized for ignoring the masses in favor of outliers and disadvantaged groups, and its inherent susceptibility to minority rule. Concretely, utilitarian aggregates only

weakly satisfy the Pigou-Dalton principle, thus do not incentivize equitable redistribution, and egalitarian aggregates satisfy only weak monotonicity, thus only incentivize gains in the *most disadvantaged* group(s). Power-means provide a spectrum of intermediaries, so exactly how tradeoffs should be made may depend on the application, as well as the culturosocietal values of the social planner. They are also *statistically convenient*, as many of our estimation guarantees hold in terms of generic Lipschitz-continuity assumptions, and thus apply to any power-mean malfare function.

Malfare of Regret. Regret measures the *relative dissatisfaction* of group i with some $h \in \mathcal{H}$, relative to their preferred $\mathbf{h}_i^* \in \mathcal{H}$. We define the (per-group) preferred outcome \mathbf{h}_i^* as the model group i would select for themselves, i.e.,

$$\mathbf{h}_i^* \doteq \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{E}_{\mathcal{D}_i} [\ell \circ h] \quad \text{or} \quad \mathbf{h}_i^* \doteq \operatorname{argmax}_{h \in \mathcal{H}} \mathbb{E}_{\mathcal{D}_i} [u \circ h] , \quad (2)$$

for loss or utility, respectively, and we let s_i^* denote the *optimal expected sentiment* for group i , i.e., $s_i^* \doteq \mathbb{E}_{\mathcal{D}_i} [s \circ \mathbf{h}_i^*]$. We now formally define the *regret* of group i on some outcome or model $h \in \mathcal{H}$ as

$$\operatorname{Reg}_i(h) \doteq \mathbb{E}_{\mathcal{D}_i} [\ell \circ h] - s_i^* , \quad \operatorname{Reg}_i(h) \doteq s_i^* - \mathbb{E}_{\mathcal{D}_i} [u \circ h] , \quad \text{or generically,} \quad \operatorname{Reg}_i(h) \doteq \left| \mathbb{E}_{\mathcal{D}_i} [s \circ h] - s_i^* \right| . \quad (3)$$

Intuitively $\operatorname{Reg}_i(h)$ is nonnegative, and it quantifies the amount by which group i prefers their optimal \mathbf{h}_i^* to h .

Several authors [7, 39] minimize the worst-case (over groups) *regret* of the selected \hat{h} , and the statistical and computational questions that arise are studied under the umbrella of “multi-group agnostic PAC learning.” We generalize this notion, optimizing not just *worst-case* (i.e., egalitarian), but *arbitrary malfare functions*, of per-group regret values, which allows for greater flexibility and resistance to the usual issues of egalitarian malfare. In particular, we seek

$$\mathbf{h}^* \doteq \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{M} \left(i \mapsto \operatorname{Reg}_i(h); \mathbf{w} \right) = \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{M} \left(i \mapsto \left| \mathbb{E}_{\mathcal{D}_i} [s \circ h] - s_i^* \right|; \mathbf{w} \right) . \quad (4)$$

Curiously, since we seek to measure overall regret, and regret is a nonnegative quantity with negative connotation, we always summarize it with a malfare function $\mathbb{M}(\cdot; \mathbf{w})$, even when we began with a *utility function*. Intuitively, this is because we can never hope to select a shared function \hat{h} that group i prefers to \mathbf{h}_i^* , thus *excess dissatisfaction* is always positive in both the loss and utility cases. In some sense, the malfare of regret thus measures the *price of sharing* in a society, as the shared model \hat{h} is naturally compared [23, 49] to letting each group select their own model \hat{h}_i .

Previous work summarizes regret across groups by taking the *largest regret* amongst them. This is analogous to game-theoretic regret (i.e., the maximum over agents of *utility differences* at *adjacent profiles*), but even there, *any malfare function* could reasonably aggregate per-group regret values. We argue that considering only *egalitarian regret* may act as an enforcer of the status quo, if one group is particularly happy with their \mathbf{h}_i^* and is thus aggrieved by any compromise — perhaps best summarized by the adage, “To those accustomed to privilege, equality feels like oppression.” We mitigate this issue by summarizing regret with a power-mean malfare function $\mathbb{M}_p(\cdot; \mathbf{w})$, instead of the egalitarian malfare, in order to lessen the impact of the most aggrieved group. In particular, this class smoothly and nonlinearly interpolates between the worst-case (egalitarian) $\mathbb{M}_\infty(\cdot; \mathbf{w})$ regret and the utilitarian $\mathbb{M}_1(\cdot; \mathbf{w})$ welfare or malfare.

Fascinatingly, we find that utilitarian regret minimization reduces to utilitarian malfare or welfare optimization, as all terms involving per-group optimal sentiment can be factored into an additive constant from these objectives; observe

$$\mathbb{M}_1(i \mapsto \operatorname{Reg}_i(h); \mathbf{w}) = \mathbb{M}_1 \left(i \mapsto \left| \mathbb{E}_{\mathcal{D}_i} [s \circ h] - s_i^* \right|; \mathbf{w} \right) = \left| \left(\sum_{i \in \mathcal{Z}} \mathbf{w}_i \mathbb{E}_{\mathcal{D}_i} [s \circ h] \right) - \mathbf{w} \cdot \mathbf{s}^* \right| = \begin{cases} s = \ell & \mathbb{M}_1(\mathbb{E}_{\mathcal{D}_i} [\ell \circ h]; \mathbf{w}) - \mathbf{w} \cdot \mathbf{s}^* \\ s = u & \mathbf{w} \cdot \mathbf{s}^* - \mathbb{W}_1(\mathbb{E}_{\mathcal{D}_i} [u \circ h]; \mathbf{w}) \end{cases} , \quad (5)$$

namely s^* appears only in the additive constant $\mathbf{w} \cdot s^*$, which is independent of h . From this perspective, we conclude that while the utilitarian regret is not particularly interesting, the power-mean malfare of regret *interpolates between* minimizing largest regret, with its minority rule issues, and optimizing utilitarian welfare or malfare.

2.2 Three Sampling Models for Populations with Multiple Groups

In order to study efficient sampling, we must first *quantify* the *cost* of a sampling-based estimation routine, which requires a *sampling model*. Within a single-group population, methods like *i.i.d. sampling*, *importance sampling*, or *sampling without replacement* are near-ubiquitous, and all can measure cost as *sample size* $m \in \mathbb{Z}_+$; however, in group-sensitive settings, we must consider how samples from different groups are obtained, and what the cost of collecting these samples is. We do not argue for a one-size-fits-all analysis, but rather we discuss three sampling models, and show that they fit key applications in the computer science domain and beyond.

1) *Joint Sampling*: Each i.i.d. sample contains a piece of information for each of the g groups, with arbitrary dependencies *between groups*. For example, per-group representatives could be shown a shared $x \in \mathcal{X}$ and asked for their feedback, which would then be used to establish some \mathcal{Y}_i for each group i . Thus each sample is in the space $\mathcal{X} \times \mathcal{Y}^g$ if the \mathcal{X} components are shared between groups, or more generally in $(\mathcal{X} \times \mathcal{Y})^g$. This setting also arises in *multi-objective reinforcement learning* [15, 44], as well as various bandit problems and empirical game theoretic analysis [48], where each query of an *action* or *strategy profile* yields a sample of the utility values of each *player*, *agent*, or *group*.

2) *Mixture Sampling*: For each sample, the data are only relevant to a nonempty *subset of groups* $z \in 2^{\mathcal{Z}}$, thus samples are elements of $\mathcal{X} \times \mathcal{Y} \times 2^{\mathcal{Z}}$. This generality is useful for studying concepts like *intersectionalism* and *multicalibration* [39], where individuals may belong to multiple groups, but the case of *mutually exclusive* groups (i.e., each sample belongs to exactly one group) is also convenient [23]. This model naturally represents a mixed population being sampled i.i.d., where the group identities of the sample are left up to chance (i.e., roughly proportional to group frequencies), and is thus the most appropriate model for learning from existing datasets [14] with *group identity features* [21].

3) *Conditional Sampling*: Here we *actively choose* from which group to sample, in contrast to the mixture sampling model, where we simply cast our net and “get what we get.” In particular, we sample i.i.d. $(\mathcal{X}, \mathcal{Y})$ pairs *conditioned on* some group $i \in \mathcal{Z}$, thus we may select sample sizes $\mathbf{m}_{1:g} \in \mathbb{Z}_+^g$ and draw a sample $(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{\mathbf{m}_1} \times \dots \times (\mathcal{X} \times \mathcal{Y})^{\mathbf{m}_g}$. This is a natural model in *active sampling* [1] and *scientific inquiry* settings, where initial results guide further study and resource expenditure, and similar conditional sampling structure arises in *stratified sampling* settings.

In mixture sampling, we generally assume *unit cost* $C = 1$ per sample, and in joint sampling, we assume *constant cost* $C > 1$ per joint-sample, as it is more expensive to set up a properly controlled joint sampling distribution. On the other hand, in conditional sampling, some groups may be more difficult or costly to study than others, so we assume a *cost model* $C_{1:g} \in \mathbb{R}_+^g$, where C_i is the per-sample cost for group i , thus the total cost of a sample with per-group sizes $\mathbf{m}_{1:g}$ is $\mathbf{m} \cdot C$. Note that the extra control of the conditional sampling model is extremely convenient and powerful, however it is generally more expensive than mixture sampling. These costs are entirely application dependent, so we take no stance on which is preferable, and rather focus on developing efficient learning algorithms under each sampling model.

3 STATISTICAL ANALYSIS AND ESTIMATION GUARANTEES

In this section, we discuss the statistics of estimating malfare and welfare functions. In particular, given a set \mathcal{Z} of g groups, we want to estimate the malfare, welfare, or regret of per-group expected sentiment of some h , i.e.,

$$\hat{M} \approx M(i \mapsto \mathbb{E}_{\mathcal{D}_i} [s \circ h]; \mathbf{w}) \quad , \quad \text{or} \quad \hat{M} \approx M(i \mapsto \text{Reg}_i(h); \mathbf{w}) \quad ,$$

where $\mathcal{D}_{1:g}$ are distributions over $\mathcal{X} \times \mathcal{Y}$, and $M(\cdot; \mathbf{w})$ generically represents some aggregator function. Estimating the expected loss or utility of one group is a well-studied sampling problem, but generalizing to the welfare, malfare, or regret of multiple groups introduces some subtleties. We start by noting that while the empirical mean is an unbiased estimator of expected utility or loss *of a single group*, in general there is no unbiased estimator of welfare or malfare (essentially due to their nonlinear nature, much like with the *standard deviation*). Thus rather than unbiased estimators, we seek *additive error* (AE) bounds of the form $\mathbb{P}(|M - \hat{M}| \leq \varepsilon) \geq 1 - \delta$, where ε is the *confidence radius* (a.k.a. the *margin of error*), and δ is the *failure probability* (or, by alternative convention, $1 - \delta$ is the *level of confidence*).

In machine learning, we optimize over a *hypothesis class* $\mathcal{H} \subseteq \mathcal{X} \rightarrow \mathcal{Y}'$, thus we seek some sample-dependent $\hat{h} \in \mathcal{H}$ with true objective value within ε of that of the optimal $h^* \in \mathcal{H}$. At times, we are also interested in related statistics, like the objective values of \hat{h} and h^* , and in general, tools to bound the deviations between the empirical and true objective values for any $h \in \mathcal{H}$ are sufficient to bound these quantities. The rest of this section pursues such bounds, assuming a fixed *failure probability* δ and *sample size* m_i for each group $i \in \mathcal{Z}$. In particular, §3.1 reviews known results for uniformly estimating expectations across \mathcal{H} , §3.2 builds upon these results to uniformly estimate malfare, welfare, and regret values, and §3.3 then studies how varying per-group sentiment values and confidence radii impacts these bounds, and quantifies the incremental value of sampling from each group as a function of these quantities.

3.1 Uniform Convergence Bounds for Mean Estimation

In this work, the common functional form of our additive error (AE) bounds is *data-dependent uniform-convergence*, vectorized to operate over samples from multiple groups, rather than on a single-group sample. Occasionally, we are interested in the scalar form $\text{AES}(m, \delta, \mathbf{x}, \mathbf{y}) : \mathbb{Z}_+ \times (0, 1) \times \mathcal{X}^m \times \mathcal{Y}^m \rightarrow \mathbb{R}_{0+}$, which operates on a single group, but unless otherwise stated, we refer to the vector bound $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) : \mathbb{Z}_+^g \times (0, 1) \times (\mathcal{X}^{m_1} \times \dots \times \mathcal{X}^{m_g}) \times (\mathcal{Y}^{m_1} \times \dots \times \mathcal{Y}^{m_g}) \rightarrow \mathbb{R}_{0+}$. In particular, given a sample $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}_1^{m_1} \times \dots \times \mathcal{D}_g^{m_g}$, we require a *function*² $\text{AEV}(\dots)$ such that

$$\hat{\varepsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) \implies \mathbb{P} \left(\max_{i \in \mathcal{Z}} \sup_{h \in \mathcal{H}} \left| \hat{\mathbb{E}}_{\mathcal{D}_i} [s \circ h] - \mathbb{E}_{\mathcal{D}_i} [s \circ h] \right| - \hat{\varepsilon}_i > 0 \right) < \delta. \quad (6)$$

The supremum term in (6), known as the *supremum deviation*, measures the *largest difference* over \mathcal{H} between empirical and true sentiment. Section 3.2 explores how $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$ can be used to bound malfare, welfare, and regret, and the remainder of this subsection is dedicated to showing nontrivial bounds of this form for machine learning applications. All of our AE bounds assume *bounded sentiment range* $r \doteq \sup_{y' \in \mathcal{Y}', y \in \mathcal{Y}} s(y', y)$, but this can usually be relaxed if we instead assume a *moment condition*, e.g., each $s \circ h$ is sub-Gaussian, sub-exponential, sub-gamma, or sub-Poisson [8].

Data-dependent uniform-convergence bounds, i.e., those of the form $\text{AES}(m, \delta, \mathbf{x}, \mathbf{y})$, are invaluable for studying a population about which little is known. Such bounds require data to evaluate, thus we cannot determine *a priori* how much data will be required to meet a given confidence radius. This contrasts distribution-free bounds, which must have worst-case dependence on the distribution, and take the form $\text{AES}(m, \delta) \leq \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^m} \text{AES}(m, \delta, \mathbf{x}, \mathbf{y})$. In §4, when constructing schedules for progressive sampling, we often assume knowledge of $\text{AES}(m, \delta)$, but this is usually possible via this worst-case RHS bound. We first present simple bounds for *bounded finite hypothesis classes*, which depend on the *sentiment range* r , hypothesis class size $|\mathcal{H}|$, variances $\mathbb{V}[\cdot]$, and empirical variances $\hat{\mathbb{V}}[\cdot]$.

² Going forward, we present only scalar bounds, but it is to be understood that given an additive error *scalar bound* $\text{AES}(\dots)$ and a finite group count g , we may construct the additive error *vector bound* $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) \leftarrow \langle \text{AES}(m_1, \frac{\delta}{g}, \mathbf{x}_{1:}, \mathbf{y}_{1:}), \dots, \text{AES}(m_g, \frac{\delta}{g}, \mathbf{x}_{g:}, \mathbf{y}_{g:}) \rangle$ via the union bound.

Theorem 3.1 (Uniform Convergence for Bounded Finite Hypothesis Classes). We may bound the distribution-free $\text{AES}(m, \delta)$, the distribution-dependent $\text{AES}_{\mathcal{D}}(m, \delta)$, and the data-dependent $\text{AES}(m, \delta, \mathbf{x}, \mathbf{y})$ scalar additive error as

$$1) \varepsilon \leftarrow \sqrt{\frac{r^2 \ln \frac{2|\mathcal{H}|}{\delta}}{2m}} \quad [27]; \quad 2) \varepsilon \leftarrow \frac{r \ln \frac{2|\mathcal{H}|}{\delta}}{3m} + \sup_{h \in \mathcal{H}} \sqrt{\frac{2 \mathbb{V}_{\mathcal{D}}[\text{s} \circ h] \ln \frac{2|\mathcal{H}|}{\delta}}{m}} \quad [6]; \quad \& \quad 3) \hat{\varepsilon} \leftarrow \frac{7r \ln \frac{2|\mathcal{H}|+1}{\delta}}{3(m-1)} + \sup_{h \in \mathcal{H}} \sqrt{\frac{2 \hat{\mathbb{V}}_{\mathbf{x}, \mathbf{y}}[\text{s} \circ h] \ln \frac{2|\mathcal{H}|+1}{\delta}}{(m-1)}} \quad [16].$$

Note that supremum variances and empirical variances are properties of the distribution and sample, respectively. Dependence on variance is necessary (similar terms appear in mean-estimation lower-bounds [19, 32]), however the $\ln|\mathcal{H}|$ union bound terms are loose, and the bounds are vacuous for infinite (continuous) \mathcal{H} . We now state results using Rademacher averages [5, 42] that tolerate infinite \mathcal{H} , while preserving the variance-dependence of item 2.

Theorem 3.2 (Uniform Convergence with Rademacher Averages). Suppose hypothesis class \mathcal{H} and sentiment function $\text{s}(\cdot, \cdot)$, take $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}^m$ and $\sigma \sim \mathcal{U}^m(\pm 1)$, i.e., σ is uniformly distributed on $(\pm 1)^m$, and define the *Rademacher average* $\mathfrak{R}_m(\text{s} \circ \mathcal{H}, \mathcal{D})$ and *Bousquet variance proxy* $\mathfrak{V}_m(\text{s} \circ \mathcal{H}, \mathcal{D})$ [see 9] as

$$\mathfrak{R}_m(\text{s} \circ \mathcal{H}, \mathcal{D}) \doteq \mathbb{E}_{\mathbf{x}, \mathbf{y}, \sigma} \left[\sup_{h \in \mathcal{H}} \left| \frac{1}{m} \sum_{i=1}^m \sigma_i (\text{s} \circ h)(x_i, y_i) \right| \right], \quad \mathfrak{V}_m(\text{s} \circ \mathcal{H}, \mathcal{D}) \doteq \sup_{h \in \mathcal{H}} \mathbb{V}[\text{s} \circ h] + 4r \mathfrak{R}_m(\text{s} \circ \mathcal{H}, \mathcal{D}). \quad (7)$$

We may then bound $\text{AES}_{\mathcal{D}}(m, \delta)$ as $\varepsilon \leftarrow 2\mathfrak{R}_m(\text{s} \circ \mathcal{H}, \mathcal{D}) + \frac{r \ln \frac{1}{\delta}}{3m} + \sqrt{\frac{2\mathfrak{V}_m(\text{s} \circ \mathcal{H}, \mathcal{D}) \ln \frac{1}{\delta}}{m}}$.

Data-dependent analogs of thm. 3.2 are possible using *empirical Rademacher averages* and *variances* at no asymptotic cost [16]. In the worst case, thm. 3.2 performs comparably to thm. 3.1 item 2, however it improves when *correlations* exist between elements of \mathcal{H} , because the *effective size* of \mathcal{H} is smaller for the purposes of realizing the supremum in the Rademacher average, see (7). The abstract inequalities of thm. 3.2 are quite opaque, so we now provide concrete bounds on the Rademacher averages of practical infinite hypothesis classes. The below results hold *for any distribution* \mathcal{D} , and are thus distribution-free, although similar distribution-dependent or data-dependent bounds are possible.

Property 3.3 (Practical Bounds on Rademacher Averages). Suppose as in thm. 3.2. The following then hold.

- 1) Suppose \mathcal{H} has Vapnik-Chervonenkis (VC) dimension d [46, 47], and $\ell(y', y) \doteq 1 - \mathbb{1}_{\{y\}}(y')$ is the 0-1 loss. Then for some absolute constant c , $\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) \leq \sqrt{\frac{cd}{m}}$, which implies bounds for linear classifiers, bounded-depth decision trees [31], and many classes of neural network [2].
- 2) Suppose $\mathcal{X} \doteq \{\vec{x} \in \mathbb{R}^\infty \mid \|\vec{x}\|_2 \leq R\}$ is the R -radius \mathcal{L}_2 ball in \mathbb{R}^∞ , $\mathcal{H} \doteq \{\vec{x} \mapsto \vec{w} \cdot \vec{x} \mid \|\vec{w}\|_2 \leq \gamma\}$ is a γ -regularized linear class, $\mathcal{Y} \doteq [-R\gamma, R\gamma]$, and $\ell(\cdot, \cdot)$ is a λ -Lipschitz loss function s.t. $\ell(y, y) = 0$. Then $r \leq 2\lambda R\gamma$ and $\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) \leq \frac{2\lambda R\gamma}{\sqrt{m}}$. This implies bounds for (kernelized) SVM, generalized linear models [35], and bounded linear regression.

3.2 From Mean Estimation to Welfare, Malfare, and Regret Bounds

We now adapt the AE bounds of §3.1 on expectations to bound malfare, welfare, and regret in terms of empirical estimates thereof. In particular, the strategy for each is to combine tail bounds for mean-estimation with the *monotonicity axiom* (def. 2.1 item 1) to bound the tails and expectations of our desiderata. We use the uniform convergence bounds of §3.1 to bound the error of these estimates, thus we need only propagate this uncertainty through the appropriate aggregator functions. In general, aggregator functions are nonlinear, and optimizing over \mathcal{H} results in estimation bias, thus the plug-in estimator is biased, however, we still obtain *tail bounds* on our objectives via $\text{AEV}(\dots)$. Because the plug-in estimator is biased, we also consider various LCB-and-UCB-style estimates, which when optimized yield safer function choices and partially control for overfitting. Finally, in some cases, integrating over worst-case uncertainty from the tail bounds of $\text{AEV}(\dots)$ yields convenient bounds on the expectation (and thus the bias) of the plug-in estimator.

Welfare and Malfare. Due to the lack of an unbiased estimator for welfare and malfare, we study the simple plug-in estimator \hat{M} , as employed by [14], and introduce a pair of lower and upper estimators $(\hat{M}^\downarrow, \hat{M}^\uparrow)$. In particular, we take

$$\hat{M} \doteq \underbrace{M\left(i \mapsto \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h]; \mathbf{w}\right)}_{\text{PLUG-IN ESTIMATE}}, \quad \hat{M}^\downarrow \doteq \underbrace{M\left(i \mapsto 0 \vee \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h] - \hat{\epsilon}_i; \mathbf{w}\right)}_{\text{LCB ESTIMATE}}, \quad \& \quad \hat{M}^\uparrow \doteq \underbrace{M\left(i \mapsto c \wedge \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h] + \hat{\epsilon}_i; \mathbf{w}\right)}_{\text{UCB ESTIMATE}}, \quad (8)$$

where \vee and \wedge are the (minimum precedence) infix binary max and min operators. By monotonicity (axiom 1), it holds that $\hat{M}^\downarrow \leq \hat{M} \leq \hat{M}^\uparrow$. The *lower* and *upper confidence bound* estimates are convenient, both to show high probability bounds, and to sandwich the plug-in estimator, which we use to bound its bias. We first show tail bounds for the estimation of welfare and malfare in terms of their plug-in, LCB, and UCB estimates, and we then bound the bias of \hat{M} .

Theorem 3.4 (Welfare and Malfare Tail Bounds). Suppose sentiment function $s(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$, per-group probability distributions $\mathcal{D}_{1:g}$, sample size vector $\mathbf{m} \in \mathbb{Z}_+^g$, samples $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}_1^{\mathbf{m}_1} \times \dots \times \mathcal{D}_g^{\mathbf{m}_g}$, failure probability $\delta \in (0, 1)$, and additive error bound $\text{AEV}(\dots)$, and let $\hat{\epsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$. Then for all $h \in \mathcal{H}$ and all monotonic aggregator functions $M(\cdot; \mathbf{w})$, it holds with probability at least $1 - \delta$ over \mathbf{x}, \mathbf{y} , and $\hat{\epsilon}$ that

$$\underbrace{M\left(i \mapsto 0 \vee \mathbb{E}_{\mathcal{D}_i} [s \circ h] - \hat{\epsilon}_i; \mathbf{w}\right)}_{\text{TRUE LB}} \leq \underbrace{M\left(i \mapsto \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h]; \mathbf{w}\right)}_{\text{PLUG-IN ESTIMATE } \hat{M}} \leq \underbrace{M\left(i \mapsto c \wedge \mathbb{E}_{\mathcal{D}_i} [s \circ h] + \hat{\epsilon}_i; \mathbf{w}\right)}_{\text{TRUE UB}}, \quad \& \quad (9)$$

$$\underbrace{M\left(i \mapsto 0 \vee \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h] - \hat{\epsilon}_i; \mathbf{w}\right)}_{\text{LCB ESTIMATE } \hat{M}^\downarrow} \leq \underbrace{M\left(i \mapsto \mathbb{E}_{\mathcal{D}_i} [s \circ h]; \mathbf{w}\right)}_{\text{TRUE AGGREGATE } M} \leq \underbrace{M\left(i \mapsto c \wedge \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h] + \hat{\epsilon}_i; \mathbf{w}\right)}_{\text{UCB ESTIMATE } \hat{M}^\uparrow}, \quad (10)$$

thus if $M(\cdot; \mathbf{w})$ is λ -Lipschitz-continuous w.r.t. some norm $\|\cdot\|_M$, we have

$$\left| \underbrace{M\left(i \mapsto \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h]; \mathbf{w}\right)}_{\text{PLUG-IN ESTIMATE}} - \underbrace{M\left(i \mapsto \mathbb{E}_{\mathcal{D}_i} [s \circ h]; \mathbf{w}\right)}_{\text{TRUE AGGREGATE}} \right| \leq \lambda \|\hat{\epsilon}\|_M. \quad (11)$$

From (10), we see that minimizing \hat{M}^\uparrow (or maximizing \hat{M}^\downarrow) is in some sense a *safe choice*, as w.h.p. we can bound the true aggregate value in terms of the UCB or LCB. This idea is reminiscent of the *sample variance penalization* algorithm of [34], wherein ERM is supplanted by minimizing an *upper-bound* on risk; in that case with variance-dependent bounds, but here the bound depends on the structure of the malfare or welfare objective at hand. It should also be noted that while the final Lipschitz form (11) is concise and convenient for all Lipschitz-continuous aggregator functions (e.g., all $p \geq 1$ power-mean malfare functions, see thm. 2.2 item 3), it can be quite loose. For example, under \pm uncertainty intervals, the egalitarian welfare $W_{-\infty}(\langle 4 \pm 1, 9 \pm 8 \rangle; \mathbf{w}) = \min(4 \pm 1, 9 \pm 8)$ must be on the interval 3 ± 2 , despite (11) giving confidence radius 8. Thus while (11) is convenient for intuition and analysis, when possible (9) or (10) should be favored.

Theorem 3.5 (Welfare and Malfare Expectation Bounds). Suppose as in thm. 3.4, and assume also that $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) = \text{AEV}(\mathbf{m}, \delta)$ is a *distribution-free* or *distribution-dependent* (but *not data-dependent*) bound. Then

$$|M - \mathbb{E}[\hat{M}]| \leq \mathbb{E}[|M - \hat{M}|] \leq \lambda \int_0^1 \|\text{AEV}(\mathbf{m}, \delta)\|_M d\delta.$$

The above theorems give general recipes for bounding tails and expectations, so for demonstrative purposes, we instantiate them with thm. 3.1 for malfare estimation. Similar bounds can be derived for learning with thm. 3.2.

Corollary 3.6 (Bernstein-Type Malfare Bounds). Suppose as in thm. 3.1, and also per-group sample size m (i.e., $\mathbf{m} = \langle m, \dots, m \rangle$) and $p \geq 1$ power-mean malfare function $\Lambda_p(\cdot; \mathbf{w})$. Now, let $\mathbf{v}_i \doteq \mathbb{E}_{\mathcal{D}_i} [s \circ h]$, and define variance proxy

v as $v \doteq (\sum_{i \in \mathcal{Z}} \mathbf{w}_i \sqrt{v_i})^2$ for $p = 1$, $v \doteq \mathbf{w} \cdot \mathbf{v}$ for $p \in (1, 2]$, or $v \doteq \|\mathbf{v}\|_\infty$ for $p > 2$. Then, for all $\delta \in (0, 1)$, we have

1) $\mathbb{P}\left(|\mathcal{M} - \hat{\mathcal{M}}| \geq \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2v \ln \frac{2g}{\delta}}{m}}\right) \leq \delta$; 2) $\mathbb{E}[|\mathcal{M} - \hat{\mathcal{M}}|] \leq \frac{r \ln(2eg)}{3m} + \sqrt{\frac{2v \ln(2eg)}{m}}$; & 3) $\mathcal{M} \leq \mathbb{E}[\hat{\mathcal{M}}] \leq \mathcal{M} + \frac{r \ln(eg)}{3m} + \sqrt{\frac{2v \ln(eg)}{m}}$.

Estimating the Malfare of Regret. Regret is difficult to bound, as it depends both on the expected sentiment of the selected \hat{h} , and also on \mathcal{H} through the (unknown) per-group optimal sentiments $s_{1:g}^*$. We thus introduce the estimators

$$\hat{s}_i \doteq \inf_{h \in \mathcal{H}} \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [\ell \circ h] , \quad \text{or} \quad \hat{s}_i \doteq \sup_{h \in \mathcal{H}} \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [u \circ h] , \quad (12)$$

for loss or utility, respectively, cf. (2). By analogy with (3), the plug-in estimator for the regret malfare minimizer is then

$$\hat{h} \doteq \operatorname{argmin}_{h \in \mathcal{H}} \mathcal{M} \left(i \mapsto \left| \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [s \circ h] - \hat{s}_i \right| ; \mathbf{w} \right) . \quad (13)$$

The following theorem bounds the difference between the true and empirical malfare of regret.

Theorem 3.7 (Regret Estimation Bounds). Suppose sentiment function $s(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$, per-group probability distributions $\mathcal{D}_{1:g}$, sample size vector $\mathbf{m} \in \mathbb{Z}_+^g$, samples $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}_1^{\mathbf{m}_1} \times \dots \times \mathcal{D}_g^{\mathbf{m}_g}$, failure probability $\delta \in (0, 1)$, and additive error bound $\text{AEV}(\dots)$, and let $\hat{\epsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$. Then for all $h \in \mathcal{H}$ and all monotonic malfare functions $\mathcal{M}(\cdot; \mathbf{w})$, it holds with probability at least $1 - \delta$ over \mathbf{x}, \mathbf{y} , and $\hat{\epsilon}$ that

$$\underbrace{\mathcal{M} \left(i \mapsto 0 \vee \left| \frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - s_i^* \right| - 2\hat{\epsilon}_i ; \mathbf{w} \right)}_{\text{TRUE REGRET MALFARE LB}} \leq \underbrace{\mathcal{M} \left(i \mapsto \left| \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [s \circ h] - \hat{s}_i \right| ; \mathbf{w} \right)}_{\text{PLUG-IN REGRET MALFARE}} \leq \underbrace{\mathcal{M} \left(i \mapsto c \wedge \left| \frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - s_i^* \right| + 2\hat{\epsilon}_i ; \mathbf{w} \right)}_{\text{TRUE REGRET MALFARE UB}} , \quad \& \quad (14)$$

$$\underbrace{\mathcal{M} \left(i \mapsto 0 \vee \left| \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [s \circ h] - \hat{s}_i \right| - 2\hat{\epsilon}_i ; \mathbf{w} \right)}_{\text{LCB ESTIMATE}} \leq \underbrace{\mathcal{M} \left(i \mapsto \left| \frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - s_i^* \right| ; \mathbf{w} \right)}_{\text{TRUE REGRET MALFARE}} \leq \underbrace{\mathcal{M} \left(i \mapsto c \wedge \left| \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [s \circ h] - \hat{s}_i \right| + 2\hat{\epsilon}_i ; \mathbf{w} \right)}_{\text{UCB ESTIMATE}} , \quad (15)$$

thus if $\mathcal{M}(\cdot; \mathbf{w})$ is λ -Lipschitz-continuous w.r.t. some norm $\|\cdot\|_{\mathcal{M}}$, we have

$$\left| \underbrace{\mathcal{M} \left(i \mapsto \left| \hat{\mathbb{E}}_{\mathbf{x}_{i:}, \mathbf{y}_{i:}} [s \circ h] - \hat{s}_i \right| ; \mathbf{w} \right)}_{\text{PLUG-IN REGRET MALFARE}} - \underbrace{\mathcal{M} \left(i \mapsto \left| \frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - s_i^* \right| ; \mathbf{w} \right)}_{\text{TRUE REGRET MALFARE}} \right| \leq 2\lambda \|\hat{\epsilon}\|_{\mathcal{M}} . \quad (16)$$

Note that similar bounds on the *expectation* of the regret plug-in estimator can be shown along the lines of cor. 3.6, *mutatis mutandis* for regret. Note also that thm. 3.7 matches thm. 3.4 up to a 2-factor attached to the confidence radius, thus in some sense regret is “about twice as difficult” to estimate as malfare or welfare.

3.3 Information Asymmetry and Where Best to Sample

An intuitive notion of fairness would suggest that we should draw equally-sized samples for each group, or perhaps samples proportional to population frequencies. If the goal is to optimize or bound welfare, malfare, or regret, such intuitive notions should be rejected, as they are critically flawed. We now discuss the ways in which samples drawn from one group or another may be more or less valuable to for the purposes of estimating or optimizing these objectives.

As a brief thought experiment, suppose that we want to estimate the egalitarian welfare of a population consisting of two groups, with expected utilities $\langle 1, 10 \rangle$ that are similarly difficult to estimate. In such a setting, nearly all sampling effort should be invested in estimating the utility of group one, as once group two is estimated to within ± 9 AE, there is *no further benefit* to improving their estimate. Thus the optimal sampling strategy depends on the true expected utilities, the difficulties of estimating utilities for each group, and the objective in question, and *in no way* resembles the naive uniform or proportional so-called “fair sampling strategies” described above. We argue that such naive strategies are

dangerous, as they introduce subtle biases and fairness issues, but the rationale for alternative sampling strategies is only apparent through the lens of sample complexity.

We now ask the questions, “Given a sample, what do we need to obtain sharper bounds?” and “How much will bounds improve with a larger sample?” We begin with a soft discussion as to why samples from different groups may contribute more or less information to an estimate, which we measure as the improvement to tail bounds that additional samples may yield. In particular, for malfare, we discuss the improvement to upper bounds, but the entire discussion can be directly translated to welfare and lower bounds in the usual manner. We then quantify these factors mathematically, and we develop these ideas further in §4.2, where they are used to adaptively choose from which group to sample.

1) *Variable estimation difficulty or overfit potential*: Often it is inherently more difficult to give bounds on the expected sentiment for some groups than for others. This can be due to differences in variances (see thm. 3.1) or in uniform convergence bounds (see thm. 3.2), and in general, occurs when $\hat{\epsilon} \leftarrow \text{AEV}(\dots)$ has $\hat{\epsilon}_i \ll \hat{\epsilon}_j$, even while $\mathbf{m}_i \approx \mathbf{m}_j$.

2) *Variable task difficulty*: Some groups may be inherently easier or harder to satisfy than others; e.g., regression and classification problems are generally easier for groups with labels that are more homogeneous, and regret varies with the optimal expected sentiment \mathbf{s}^* . This is crucial, because most malfare and welfare functions are more sensitive³ to high-risk or low-utility groups, thus the ease of satisfying a group effects their impact on malfare and welfare values.

3) *Aggregator function interactions*: Complicated interactions also occur through the malfare or welfare function. When learning over \mathcal{H} , the set of near-optimal functions is more relevant than those that are clearly bad choices overall, and groups that tend to be mutually satisfied (i.e., are correlated) are less impactful to the overall objective. Weight values in malfare or welfare functions may also differ between groups, and higher-weighted groups are usually more impactful.

Quantifying the Incremental Value of Sampling. We measure the impact of sampling by asking the question, “What is the incremental value of a single sample drawn for some group?” In particular, we quantify the value of the sample as the *reduction in uncertainty*, as measured by the infimum UCB (over \mathcal{H}), and although this is inherently a discrete question, we approximate the answer for the power-mean malfare with tools from the calculus of infinitesimals.

Note that all such analysis is necessarily heuristic, as we fundamentally cannot answer this question without more information: it is precisely because we are trying to *estimate unknown means* that we can’t know how the samples we draw will impact the *empirical means*. For now, we heuristically assume that our estimated expectations are reasonably accurate, and consider what will happen as tail bounds sharpen with additional samples. The strategy we thus employ is to make a reasonable guess as to how sampling might impact the UCB by assuming that the empirical mean will not be strongly affected, and all confidence intervals over m samples will contract at a $\Theta \sqrt{\frac{1}{m}}$ rate.

Property 3.8 (Incremental Gain of Sampling). Suppose power-mean malfare $\Lambda_p(\cdot; \mathbf{w})$, sample (\mathbf{x}, \mathbf{y}) with group sample sizes $\mathbf{m}_{1:g}$, and let \mathbf{x}', \mathbf{y}' extend \mathbf{x}, \mathbf{y} to sample sizes \mathbf{m}' , where $\mathbf{m}' = \mathbf{m} + \mathbb{1}_i$, i.e., group i has one additional sample. Now, let $\hat{\epsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$ and $\tilde{\epsilon} \leftarrow \text{AEV}(\mathbf{m}', \delta, \mathbf{x}', \mathbf{y}')$, and take $\hat{h} \doteq \text{argmin}_{h \in \mathcal{H}} \Lambda_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i}[\ell \circ h] + \hat{\epsilon}_i; \mathbf{w})$, $\hat{\Lambda} \doteq \Lambda_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i}[\ell \circ \hat{h}]; \mathbf{w})$, $\hat{\Lambda}^\dagger \doteq \Lambda_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i}[\ell \circ \hat{h}] + \hat{\epsilon}_i; \mathbf{w})$, and $\tilde{\Lambda}^\dagger \doteq \inf_{h \in \mathcal{H}} \Lambda_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}'_i, \mathbf{y}'_i}[\ell \circ h] + \tilde{\epsilon}_i; \mathbf{w})$.

Then the *incremental impact* of sampling from group i on the UCB is approximately

$$\hat{\Lambda}^\dagger - \tilde{\Lambda}^\dagger \approx \frac{\hat{\epsilon}_i \mathbf{w}_i}{2\mathbf{m}_i + \frac{3}{2}} \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i}[\ell \circ \hat{h}] + \hat{\epsilon}_i}{\hat{\Lambda}^\dagger} \right)^{p-1} \approx \frac{\hat{\epsilon}_i \mathbf{w}_i}{2\mathbf{m}_i} \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i}[\ell \circ \hat{h}]}{\hat{\Lambda}} \right)^{p-1}. \quad (17)$$

Observe that (17) characterizes the knowledge gain of sampling from group i . This gain is *proportional* to the current bound radius $\hat{\epsilon}_i$, the group weight \mathbf{w}_i , and the $(p-1)$ th power of the ratio of the UCB risk of group i to the UCB malfare,

³Note that this holds for all $p \neq 1$ power-means, and is axiomatically codified by the *Pigou-Dalton transfer principle* (def. 2.1 item 7).

i.e., $(\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\epsilon}_i / \hat{M}^\dagger)^{p-1}$, and *inversely proportional* to the amount of effort \mathbf{m}_i already put forth into studying group i . These terms line up with the soft arguments at the top of §3.3 as to where sampling should occur, but it is only via precisely studying sample complexity and estimation error that we gain quantifiable mathematical insight. In particular, the weight term \mathbf{w}_i appears directly, and $\frac{\hat{\epsilon}_i}{\mathbf{m}_i}$ captures both the difficulty of estimating this group, and also the diminishing incremental improvement produced by further sampling. The ratio between the risk of group i and the malfare then captures how important group i is relative to the other groups, and this term being raised to the $(p - 1)$ th power nonlinearly adjusts its impact; higher p saturate high-risk groups, tending towards egalitarianism, whereas in the $p = 1$ (utilitarian) case, this term is 1. Finally, for optimization problems, the dependence on \hat{h} captures other dependencies; namely the behavior of $\Lambda(\cdot; \mathbf{w})$ near the optimal $h \in \mathcal{H}$ is what matters.

This analysis parallels concerns in *stratified sampling*, wherein subpopulations are sampled individually, generally to produce an improved mean estimator. In particular, we suggest a form of *disproportionate allocation*, i.e., per-group sample sizes are not necessarily proportional to their population frequencies. Rather than simply considering *variances* to estimate *means*, we holistically consider the objective and uncertainty over various quantities, thus our sample-size selection-strategy is a variant of the *minimax sampling ratio* [41] method. Chen et al. [12] also suggest disproportionate allocation in fair learning, albeit only for bounding differences of per-group statistics. Similar concerns also arise in *optimizing* minimax-fair models, wherein Abernethy et al. [1] present an algorithm that takes gradient steps to improve a model for the highest-risk group, though it is unclear whether such methods generalize beyond the egalitarian case.

4 PROGRESSIVE AND ACTIVE SAMPLING ALGORITHMS

Section 3 considers *fixed sample sizes* $\mathbf{m}_{1:g}$ and *failure probabilities* δ , and bounds the *confidence radius* ϵ . In this section, we want a fixed ϵ - δ AE guarantee, but we are willing to let an algorithm select the sample size m (or per-group sample sizes $\mathbf{m}_{1:g}$). In particular, due to the cost of sampling and processing data, we want our algorithm to minimize m (or cost measured as some function of \mathbf{m}), while constraining ϵ and δ to user-supplied levels. Some cases are simpler than others; the *joint sampling model* yields a standard progressive sampling method with a fixed sampling schedule, and the method under *mixture sampling* is similar, except a subtle conditioning argument allows us to use variably-sized per-group sample sizes based on the order in which groups are sampled. For the *conditional sampling model*, we develop an *active sampling* approach, which makes cost-sensitive decisions as to which group to sample at each iteration. More details on sampling schedules and other aspects of our progressive sampling algorithms are given in appx. B.

We can’t simply draw samples one-by-one, compute bounds using $\hat{\epsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$ after each sample, and terminate when a sufficiently sharp bound is available, because the possibility of *early termination* leads to the *multiple comparisons problem*, wherein by chance the desired confidence radius is met at some timestep. Progressive sampling algorithms correct for this by establishing a *sampling schedule* S and *failure probability schedule* Δ , which usually dictate that, at timestep t , we take a tail-bound with $\delta = \Delta_t$ and sample size S_t , while ensuring that all bounds hold simultaneously (by union bound) with probability at least $1 - \delta$. Due to this union bound, it is *inefficient* to take bounds after drawing every sample. Furthermore, for technical reasons, we henceforth assume a few mild regularity conditions:

- 1) The *sampling schedule* $S \in \mathbb{Z}_+^\infty$ is a strictly monotonically increasing sequence, i.e., for all $t \in \mathbb{Z}_+$, $S_t \leq S_{t+1}$;
- 2) The *failure probability schedule* $\Delta \in [0, 1]^\infty$ is a sequence that sums to some $\delta \in (0, 1)$, i.e., $\sum_{t=1}^\infty \Delta_t = \|\Delta\|_1 = \delta$; &
- 3) The *distribution-free bound* $\sup_{\mathbf{x}, \mathbf{y}} \|\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})\|$ is monotonically decreasing in $\mathbf{m}_{1:g}$ and δ for any norm $\|\cdot\|$.

In order to prove that a progressive sampling algorithm produces a (probabilistically) correct answer, it is crucial to show that it does not loop indefinitely. We now introduce ϵ -convergent schedules, which require all sentiment values to eventually be ϵ - δ estimated w.r.t. some norm $\|\cdot\|_M$, yielding welfare, malfare, or regret bounds via thms. 3.4 and 3.7.

Definition 4.1 (ε -Uniformly-Convergent Schedule). For any $\varepsilon \geq 0$, a sampling schedule S and failure probability schedule Δ are ε -uniformly-convergent w.r.t. $\text{AEV}(\dots)$ and some norm $\|\cdot\|_M$ if

$$\inf_{t \in \mathbb{Z}_+} \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{S_t \times g}} \|\text{AEV}(\langle S_t, \dots, S_t \rangle, \Delta_t, \mathbf{x}, \mathbf{y})\|_M \leq \varepsilon . \quad (18)$$

Intuitively, def. 4.1 captures the idea that no matter how unlucky we are with the sampled \mathbf{x}, \mathbf{y} , if $\text{AEV}(\dots)$ bounds tails once for each timestep t of the schedule, with per-group samples of size at least S_t and failure probability Δ_t , then at some point an ε -estimate of the objective will be produced. Note that neither data-dependent $\text{AEV}(\dots)$ bounds on sentiment values, nor sufficient per-group error radii to estimate the objective, are known *a priori*, thus it is not always possible to select a *sufficient* static sample size, however, def. 4.1 is *more flexible*, as it requires *only the existence* of a (possibly unknown) sufficient sample size. Even when a sufficient sample size is known, unless it is also *necessary*, progressive sampling is usually more sample-efficient, often terminating closer to the *necessary sample size*.

With this definition in hand, we now construct finite ε -, and infinite 0-, uniformly-convergent schedules. In the context of this work (see thms. 4.5 and 4.6), the finite schedule can be employed with a Lipschitz-continuous objective and an *a priori* known distribution-free bound on $\text{AEV}(\dots)$, and when the objective is continuous, but not Lipschitz-continuous, or the class \mathcal{H} is uniformly-convergent, but at an unknown rate, the infinite schedule can still be used. Both employ geometrically-increasing sample sizes, which are efficient because they never “overshoot” any sample size by more than a constant factor, while covering a range of sample sizes that is *exponential* in the number of timesteps.

Definition 4.2 (Geometric-Uniform Schedule). Suppose *optimistic size* $\alpha \geq 1$, *common ratio* $\beta > 1$, and *schedule length* $T \in \mathbb{Z}_+$. The geometric-uniform schedule then takes (geometric) $S_t \doteq \lceil \alpha \beta^t \rceil$ and (uniform) $\Delta_t \doteq \frac{\delta}{T} \mathbb{1}_{1, \dots, T}(t)$.

Definition 4.3 (Double-Geometric Schedule). Suppose *optimistic size* $\alpha > 0$ and *common ratio* $\beta > 1$. The double-geometric schedule then takes (geometric) $S_t \doteq \lceil \alpha \beta^t \rceil$ and (geometric) $\Delta_t \doteq \frac{\delta(\beta-1)}{\beta^t}$.

Lemma 4.4 (Sufficient Conditions for Uniformly-Convergent Geometric Schedules). Suppose as in def. 4.2, and assume

$$\sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{S_T \times g}} \|\text{AEV}(\langle S_T, \dots, S_T \rangle, \frac{\delta}{T}, \mathbf{x}, \mathbf{y})\|_M \leq \varepsilon . \quad (19)$$

Then the geometric-uniform schedule (S, Δ) is ε -uniformly-convergent. Furthermore, suppose as in def. 4.3, $\alpha \geq \frac{1}{\delta}$ and

$$\lim_{m \rightarrow \infty} \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{m \times g}} \|\text{AEV}(\langle m, \dots, m \rangle, \frac{\beta-1}{\beta(m+1)}, \mathbf{x}, \mathbf{y})\|_M = 0 . \quad (20)$$

Then the double-geometric schedule (S, Δ) is 0-uniformly-convergent.

The initial and final sample sizes of the geometric-uniform schedule are $S_1 = \lceil \alpha \beta \rceil$ and $S_T = \lceil \alpha \beta^T \rceil$, and often one can set S_1/β and S_T to *minimal sufficient* and *maximal necessary* sample sizes (as a function of T , the objective, and other parameters). To maximize *statistical efficiency* while controlling the value of β , we may select the minimal T such that $\lceil \log_\beta \frac{S_T}{S_1} \rceil = T$.⁴ In particular, assuming a λ -Lipschitz objective, the Hoeffding (item 1) and empirical Bernstein (item 3) bounds of thm. 3.1 imply ε -uniformly convergent schedules via (19) of length $T \in \Theta(\log \frac{\lambda T}{\varepsilon})$. For the double-geometric schedule, we may similarly set S_1/β to a minimal sufficient sample size, and here there is no T parameter (the schedule is infinite), thus we may simply select β as desired. This yields 0-uniformly convergent schedules, since each of the bounds of thm. 3.1 satisfy (20), as do those of thm. 3.2, so long as $\lim_{m \rightarrow \infty} \max_{i \in \mathcal{Z}} \mathfrak{R}_m(s \circ \mathcal{H}, \mathcal{D}_i) = 0$.

Both of the above schedule types are *efficient*, in the sense that for the smallest (per-group) static sample size m^* at which we obtain the bound ε^* , some $\hat{m} \leq \lceil \beta m^* \rceil$ is contained in the schedule, and the bound $\hat{\varepsilon} \leftarrow \text{AEV}(\langle \hat{m}, \dots, \hat{m} \rangle, \dots)$ exceeds ε^* only because it uses a smaller δ value. In particular, assuming all bounds are $\Theta\sqrt{u}$ for $u \doteq \ln \frac{g}{\delta}$, we have for each group i that $\frac{\varepsilon_i^*}{\hat{\varepsilon}_i} \in \Theta\sqrt{\frac{u}{\log(T)+u}}$ and $\frac{\varepsilon_i^*}{\hat{\varepsilon}_i} \in \Theta\sqrt{\frac{u}{\log(m^*)+u}}$ for the geometric-uniform and double-geometric

⁴The base- β logarithm arises intuitively, as the *number of times* the sample size must increase by a factor β to reach size S_T from S_1 .

Algorithm 1 Fair Learning with Linear Progressive Sampling under the Joint and Mixture Sampling Models

```

1: procedure LINEARPSLOSS( $\mathcal{H}, \ell(\cdot, \cdot), \mathcal{D}, \text{AEV}(\dots), \mathbf{S}, \Delta, \varepsilon, \mathbb{M}(\cdot; \mathbf{w}), \text{REG}$ )  $\rightarrow (\hat{h}, \hat{\mu}, \hat{\varepsilon}, \mathbb{M}^{*\downarrow})$ 
2: input: Hypothesis class  $\mathcal{H} \subseteq \mathcal{X} \rightarrow \mathcal{Y}'$ , loss function  $\ell(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow [0, c]$ , joint or mixture distribution  $\mathcal{D}$ , additive error vector bound  $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$ , schedule  $\mathbf{S} \in \mathbb{Z}_+^\infty$  and  $\Delta \in [0, 1)^\infty$ , confidence radius  $\varepsilon$ , weighted malfare  $\mathbb{M}(\cdot; \mathbf{w})$ , and Boolean REG
3: output: Empirically UCB-optimal  $\hat{h}$ , empirical malfare estimate  $\hat{\mu}$ , confidence radius  $\hat{\varepsilon}$ , and lower bound on minimal malfare  $\mathbb{M}^{*\downarrow}$ 
4:  $\mathbf{m}_{1:g} \leftarrow \mathbf{0}$ ;  $\mathbf{x}_{1:g} \leftarrow \langle \langle \cdot \rangle, \dots, \langle \cdot \rangle \rangle$ ;  $\mathbf{y}_{1:g} \leftarrow \langle \langle \cdot \rangle, \dots, \langle \cdot \rangle \rangle$   $\triangleright$  Initialize per-group sample counts, empty per-group sample lists
5: for  $t \in 1, 2, \dots$  do  $\triangleright$  Progressive sampling timesteps
6:   if  $\mathcal{D}$  is joint sampler then
7:      $(\mathbf{x}_{1:g, S_{t-1}+1:S_t}, \mathbf{y}_{1:g, S_{t-1}+1:S_t}) \sim \mathcal{D}^{\mathbf{S}_t - \mathbf{S}_{t-1}}$ ;  $\forall i \in \mathcal{Z} : \mathbf{m}_i \leftarrow S_t$   $\triangleright$  Sample from joint distribution (assume  $\mathbf{S}_0 = 0$ )
8:   else if  $\mathcal{D}$  is mixture sampler then
9:     while  $\min_{i \in \mathcal{Z}} \mathbf{m}_i < S_t$  do
10:       $(x, y, z) \sim \mathcal{D}$   $\triangleright$  Draw  $\mathcal{X} \times \mathcal{Y} \times 2^{\mathcal{Z}}$  triplet (domain, codomain, groups)
11:       $\forall i \in z : \mathbf{m}_i \leftarrow \mathbf{m}_i + 1$ ;  $(\mathbf{x}_{i, \mathbf{m}_i}, \mathbf{y}_{i, \mathbf{m}_i}) \leftarrow (x, y)$   $\triangleright$  Increment count  $\mathbf{m}_i$  and store sample for each group  $i$  associated with  $(x, y)$ 
12:    end while
13:   end if
14:    $\hat{\varepsilon}_{1:g} \leftarrow (1 + \mathbb{1}_{\text{REG}}) \text{AEV}(\mathbf{m}, \Delta_t, \mathbf{x}, \mathbf{y})$   $\triangleright$  Bound additive error of per-group supremum deviations (w.h.p.)
15:    $\forall i \in \mathcal{Z} : \hat{s}_i^* \leftarrow (\inf_{h \in \mathcal{H}} \hat{\mathbb{E}}_{\mathbf{x}_{i, \cdot}, \mathbf{y}_{i, \cdot}} [\ell \circ h])$  if REG else 0  $\triangleright$  Estimate regret baseline as per-group minimal empirical risks (if applicable)
16:    $\hat{h} \leftarrow \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{M}(i \mapsto c \wedge \hat{\mathbb{E}}_{\mathbf{x}_{i, \cdot}, \mathbf{y}_{i, \cdot}} [\ell \circ h] - \hat{s}_i^* + \hat{\varepsilon}_i; \mathbf{w})$   $\triangleright$  Compute UCB-optimal  $\hat{h}$ 
17:    $\mathbb{M}^{*\downarrow} \leftarrow \inf_{h \in \mathcal{H}} \mathbb{M}(i \mapsto 0 \vee \hat{\mathbb{E}}_{\mathbf{x}_{i, \cdot}, \mathbf{y}_{i, \cdot}} [\ell \circ h] - \hat{s}_i^* - \hat{\varepsilon}_i; \mathbf{w})$   $\triangleright$  Lower-bound optimal  $\mathbb{M}^*$ 
18:    $(\hat{\mathbb{M}}^\downarrow, \hat{\mathbb{M}}^\uparrow) \leftarrow (\mathbb{M}(i \mapsto 0 \vee \hat{\mathbb{E}}_{\mathbf{x}_{i, \cdot}, \mathbf{y}_{i, \cdot}} [\ell \circ \hat{h}] - \hat{s}_i^* - \hat{\varepsilon}_i; \mathbf{w}), \mathbb{M}(i \mapsto c \wedge \hat{\mathbb{E}}_{\mathbf{x}_{i, \cdot}, \mathbf{y}_{i, \cdot}} [\ell \circ \hat{h}] - \hat{s}_i^* + \hat{\varepsilon}_i; \mathbf{w}))$   $\triangleright$  LCB and UCB on  $\hat{h}$  (regret) malfare
19:   if  $\hat{\mathbb{M}}^\uparrow \leq \mathbb{M}^{*\downarrow} + 2\varepsilon$  then  $\triangleright$  Check if desired error guarantee is met (termination condition)
20:      $(\hat{\mu}, \hat{\varepsilon}) \leftarrow (\frac{1}{2}(\hat{\mathbb{M}}^\downarrow + \hat{\mathbb{M}}^\uparrow), \frac{1}{2}(\hat{\mathbb{M}}^\uparrow - \hat{\mathbb{M}}^\downarrow))$   $\triangleright$  Symmetric estimate  $\hat{\mu}$  and confidence radius  $\hat{\varepsilon}$  of (regret) malfare of  $\hat{h}$ 
21:     return  $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, \mathbb{M}^{*\downarrow})$   $\triangleright$  Return UCB-optimal  $\hat{h}$ ,  $\hat{\varepsilon}$ -estimate of  $\mathbb{M}(\cdot; \mathbf{w})$ , and lower-bound on optimal malfare  $\mathbb{M}^{*\downarrow}$ 
22:   end if
23: end for
24: end procedure
25: procedure LINEARPSUTILITY( $\mathcal{H}, u(\cdot, \cdot), \mathcal{D}, \text{AEV}(\dots), \mathbf{S}, \Delta, \varepsilon, \mathbb{M}(\cdot; \mathbf{w}), \text{REG}$ )  $\rightarrow (\hat{h}, \hat{\mu}, \hat{\varepsilon}, \mathbb{M}^{*\uparrow})$ 
26: input: Utility function  $u(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow [0, c]$ , weighted aggregator function  $\mathbb{M}(\cdot; \mathbf{w})$  (malfare if REG, otherwise welfare), see line 2
27: output: Empirically LCB-optimal  $\hat{h}$ , empirical welfare  $\hat{\mu}$ , confidence radius  $\hat{\varepsilon}$ , and UB on maximal welfare  $\mathbb{M}^{*\uparrow}$  (or similar for regret)
28:  $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, \mathbb{M}^{*\uparrow}) \leftarrow \text{LINEARPSLOSS}(\mathcal{H}, c - u(\cdot, \cdot), \mathcal{D}, \text{AEV}(\dots), \mathbf{S}, \Delta, \varepsilon, (2\mathbb{1}_{\text{REG}} - 1)\mathbb{M}(\cdot; \mathbf{w}), \text{REG})$   $\triangleright$  Negate to flip inf and sup
29: return  $(\hat{h}, c - \hat{\mu}, \hat{\varepsilon}, (2\mathbb{1}_{\text{REG}} - 1)\mathbb{M}^{*\uparrow})$   $\triangleright$  Invert negation (as applicable)
30: end procedure

```

schedules, respectively. Note also that $\log(T) \in \Theta(\log \log \frac{r\lambda}{\varepsilon})$, whereas $\log(m^*) \in \mathcal{O}(\log \frac{r\lambda u}{\varepsilon})$, thus the geometric-uniform schedule is preferable, unless m^* is exponentially smaller than the above bound, e.g., if $\lambda = \infty$, or if a nonlinear objective is more stable to perturbations of each s_i about its optimum than the Lipschitz constant λ would indicate.

4.1 The Linear Progressive Sampling Algorithm

The core of *linear progressive sampling* (alg. 1) is quite simple. At timestep $t = 1$, we *guess* that a sample of size \mathbf{S}_1 for all groups will be sufficient to ε - δ optimize the objective, we draw at least such a sample (line 7 for joint sampling, or lines 9–12 for mixture sampling), compute tail bounds (line 14), then determine the UCB-optimal \hat{h} (line 16). If our bounds indicate that \hat{h} is provably near-optimal, alg. 1 terminates, otherwise, our guess was incorrect, so we increment t , draw at least \mathbf{S}_t samples (per-group), and repeat. The basic principle is quite flexible, so alg. 1 can maximize welfare or minimize malfare of risk or regret via the `LINEARPSLOSS(...)` and `LINEARPSUTILITY(...)` routines.

Theorem 4.5 shows that alg. 1 learns an optimal $h \in \mathcal{H}$ to within user-specified ε - δ AE. We require only monotonicity (axiom 1) and continuity (axiom 3) of $\mathbb{M}(\cdot; \mathbf{w})$, though the power-mean malfare family is convenient, as Lipschitz-continuity (thm. 2.2 item 3) permits efficient ε -uniformly-convergent schedules (def. 4.2). NB this result generalizes to welfare objectives, *mutatis mutandis* (flipping infima and suprema), via the negation reduction of lines 25–30.

Algorithm 2 Fair Learning with Braided Progressive Sampling under the Conditional Sampling Model

```

1: procedure BRAIDEDPSLOSS( $\mathcal{H}, \ell(\cdot, \cdot), \mathcal{D}_{1:g}, C_{1:g}, \text{AES}(\dots), S, \Delta, \varepsilon, \mathcal{M}(\cdot; \mathbf{w}), \text{REG}) \rightarrow (\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{*\downarrow})$ 
2: input: Hypothesis class  $\mathcal{H}$ , loss function  $\ell(\cdot, \cdot)$ , per-group distributions  $\mathcal{D}_{1:g}$ , cost model  $C_{1:g} \in \mathbb{R}_+^g$ , additive error scalar bound  $\text{AES}(m, \delta, \mathbf{x}, \mathbf{y})$ , schedule  $S \in \mathbb{Z}_+^\infty$  and  $\Delta \in [0, 1)^\infty$ , confidence radius  $\varepsilon$ , weighted malfare  $\mathcal{M}(\cdot; \mathbf{w})$ , and Boolean  $\text{REG}$ 
3: output: Empirically UCB-optimal  $\hat{h}$ , empirical malfare estimate  $\hat{\mu}$ , confidence radius  $\hat{\varepsilon}$ , and lower bound on minimal malfare  $M^{*\downarrow}$ 
4:  $t_{1:g} \leftarrow 1$  ▷ Initialize per-group timestep indices
5:  $\forall i \in \mathcal{Z} : (\mathbf{x}_{i,1:S_i}, \mathbf{y}_{i,1:S_i}) \sim \mathcal{D}_i^{S_i}; \hat{\varepsilon}_i \leftarrow (1 + \mathbb{1}_{\text{REG}}) \text{AES}(S_i, \frac{\Delta_i}{g}, \mathbf{x}_{i,:}, \mathbf{y}_{i,:})$  ▷ Draw initial sample for all groups & bound error
6: loop ▷ Main loop over braided algorithm iterations
7:  $\forall i \in \mathcal{Z} : \hat{s}_i^* \leftarrow \left( \inf_{h \in \mathcal{H}} \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [\ell \circ h] \right)$  if  $\text{REG}$  else 0 ▷ Estimate regret baseline as per-group minimal empirical risks (if applicable)
8:  $\hat{h} \leftarrow \underset{h \in \mathcal{H}}{\text{argmin}} \mathcal{M}(i \mapsto c \wedge \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [\ell \circ h] - \hat{s}_i^* + \hat{\varepsilon}_i; \mathbf{w})$  ▷ Compute UCB-optimal  $\hat{h}$ 
9:  $M^{*\downarrow} \leftarrow \inf_{h \in \mathcal{H}} \mathcal{M}(i \mapsto 0 \vee \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [\ell \circ h] - \hat{s}_i^* - \hat{\varepsilon}_i; \mathbf{w})$  ▷ Lower-bound optimal  $M^*$ 
10:  $(\hat{M}^\downarrow, \hat{M}^\uparrow) \leftarrow \left( \mathcal{M}(i \mapsto 0 \vee \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [\ell \circ \hat{h}] - \hat{s}_i^* - \hat{\varepsilon}_i; \mathbf{w}), \mathcal{M}(i \mapsto c \wedge \mathbb{E}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [\ell \circ \hat{h}] - \hat{s}_i^* + \hat{\varepsilon}_i; \mathbf{w}) \right)$  ▷ LCB and UCB on  $\hat{h}$  (regret) malfare
11: if  $\hat{M}^\uparrow \leq M^{*\downarrow} + 2\varepsilon$  then ▷ Check if desired error guarantee is met (termination condition)
12:  $(\hat{\mu}, \hat{\varepsilon}) \leftarrow \left( \frac{1}{2}(\hat{M}^\downarrow + \hat{M}^\uparrow), \frac{1}{2}(\hat{M}^\uparrow - \hat{M}^\downarrow) \right)$  ▷ Symmetric estimate of  $\hat{\mu}$  of malfare or regret of  $\hat{h}$ 
13: return  $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{*\downarrow})$ 
14: end if
15:  $\forall i, j \in \mathcal{Z}, t \in \mathbb{Z}_+ : \tilde{\varepsilon}_{j,t}^{(i)} \leftarrow \left( \hat{\varepsilon}_j \text{ if } i \neq j \text{ else } \hat{\varepsilon}_j \sqrt{\frac{S_{t_j} \ln \frac{g}{\Delta_{t_j}}}{S_{t+t_j} \ln \frac{g}{\Delta_{t+t_j}}}} \right)$  ▷ Estimate of  $\hat{\varepsilon}_j$  after sampling group  $i$  for  $t$  more iterations
16:  $i \leftarrow \underset{i \in \mathcal{Z}}{\text{argmax}} \sup_{t \in \mathbb{Z}_+} \frac{1}{C_i(S_{t+t_i} - S_{t_i})} \left( \hat{M}^\uparrow - \mathcal{M}(j \mapsto c \wedge \mathbb{E}_{\mathbf{x}_{j,:}, \mathbf{y}_{j,:}} [\ell \circ h] - \hat{s}_j^* + \tilde{\varepsilon}_{j,t}^{(i)}; \mathbf{w}) \right)$  ▷ Select  $i$  to maximize improvement:cost ratio

RECIPROCAL COST
ESTIMATED (REGRET) MALFARE IMPROVEMENT

17:  $(\mathbf{x}_{i,1+S_{t_i}:S_{t_i+t_t}}, \mathbf{y}_{i,1+S_{t_i}:S_{t_i+t_t}}) \sim \mathcal{D}_i^{S_{t_i+t_t}-S_{t_i}}; t_i \leftarrow t_i + 1; \hat{\varepsilon}_i \leftarrow (1 + \mathbb{1}_{\text{REG}}) \text{AES}(S_{t_i}, \frac{\Delta_{t_i}}{g}, \mathbf{x}_{i,:}, \mathbf{y}_{i,:})$  ▷ Sample group  $i$  & bound error
18: end loop
19: end procedure

```

Theorem 4.5 (Linear PS Guarantees). Suppose $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{*\downarrow}) \leftarrow \text{LINEARPSLOSS}(\mathcal{H}, \ell(\cdot, \cdot), \mathcal{D}, \text{AEV}(\dots), S, \Delta, \varepsilon, \mathcal{M}(\cdot; \mathbf{w}), \text{REG})$, $M(\mathbf{s}; \mathbf{w})$ is continuous and monotonic in \mathbf{s} with (possibly infinite) Lipschitz constant λ_M w.r.t. $\|\cdot\|_M$, and the schedules (S, Δ) are $\frac{\varepsilon}{\lambda_M(1+\mathbb{1}_{\text{REG}})}$ -uniformly-convergent w.r.t. $\text{AEV}(\dots)$ and $\|\cdot\|_M$. Now take μ to be the true objective value of \hat{h} and $\mu^* \doteq \inf_{h \in \mathcal{H}} M(i \mapsto \mathbb{E}_{\mathcal{D}_i}[\ell \circ h]; \mathbf{w})$ and $\mu^* \doteq \inf_{h \in \mathcal{H}} M(i \mapsto \text{Reg}_i(h); \mathbf{w})$, or if $\text{REG} = \text{TRUE}$, take (see equation 3) $\mu \doteq M(i \mapsto \text{Reg}_i(\hat{h}); \mathbf{w})$ and $\mu^* \doteq \inf_{h \in \mathcal{H}} M(i \mapsto \text{Reg}_i(h); \mathbf{w})$. Then, with probability at least $1 - \delta$, the output $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{*\downarrow})$ obeys

$$1) |\hat{\mu} - \mu| \leq \hat{\varepsilon} \leq \varepsilon; \quad \& \quad 2) M^{*\downarrow} \leq \mu^* \leq \mu \leq \hat{\mu} + \hat{\varepsilon} \leq M^{*\downarrow} + 2\varepsilon.$$

4.2 The Braided Progressive Sampling Algorithm

Under the joint and mixture sampling models (alg. 1), progress is *linear* (i.e., *sequential*, as no decisions are made except when to terminate); we begin with (at least) S_1 samples per group, and advance until we reach a sufficient sample size to terminate with the desired guarantee. For the conditional sampling model, we present *braided progressive sampling* (alg. 2), which is *actively making decisions*, thus linear analysis is not applicable. At each *iteration* (line 6) of alg. 2, a group $i \in \mathcal{Z}$ is chosen (line 16) to optimize an estimate of knowledge-gain via logic similar to that of §3.3 (details are deferred to appx. B.2), and group i is sampled for *one additional timestep* (line 17), i.e., the sample associated with group i is extended from size S_{t_i} to S_{1+t_i} , where t_i denotes the current timestep for group i . The remainder of alg. 2 is essentially the same as alg. 1; after sampling, we optimize (line 8) a UCB-optimal \hat{h} , bound the objective (lines 9–10), and terminate if the user-supplied guarantee is met, otherwise we continue.

There is thus a *lattice* of possible sample size vectors \mathbf{m} . To avoid a union bound over this (exponentially large) lattice, we analyze the method as a *braid*, in that g progressive sampling sequences are *concurrently active*, and at each iteration

we select some group i , and advance the schedule by one timestep for only group i . Consequently, we must use (line 17) the additive error *scalar bound* $\hat{\epsilon}_i \leftarrow \text{AES}(\mathbf{m}_i, \frac{\Delta t_i}{g}, \mathbf{x}_{i,:}, \mathbf{y}_{i,:})$, i.e., we operate on *one group at a time*, rather than over all groups as in the linear algorithm (alg. 1 line 14). Similar analysis is employed for *multi-armed bandits*, where a union bound is taken over all timesteps and each arm being sampled. We now show correctness of alg. 2.

Theorem 4.6 (Braided PS Guarantees). Suppose $(\hat{h}, \hat{\mu}, \hat{\epsilon}, \mathbf{M}^{\star\downarrow}) \leftarrow \text{BRAIDEDPSLOSS}(\mathcal{H}, \ell(\cdot, \cdot), \mathcal{D}, \text{AES}(\dots), \mathbf{S}, \Delta, \epsilon, \mathbf{M}(\cdot; \mathbf{w}), \text{REG})$, $\mathbf{M}(\mathbf{s}; \mathbf{w})$ is continuous and strictly monotonic in \mathbf{s} with (possibly infinite) Lipschitz constant $\lambda_{\mathbf{M}}$ w.r.t. $\|\cdot\|_{\mathbf{M}}$, and the schedules (\mathbf{S}, Δ) are $\frac{\epsilon}{\lambda_{\mathbf{M}}(1+\mathbb{1}_{\text{REG}})}$ -uniformly-convergent w.r.t. $\|\cdot\|_{\mathbf{M}}$ and the *additive error vector bound* $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) \leftarrow \langle \text{AES}(\mathbf{m}_1, \frac{\delta}{g}, \mathbf{x}_1, \mathbf{y}_1), \dots, \text{AES}(\mathbf{m}_g, \frac{\delta}{g}, \mathbf{x}_g, \mathbf{y}_g) \rangle$. Now take μ to be the true objective value of \hat{h} and μ^* to be the true objective value of the optimal h^* (see thm. 4.5). Then, with probability at least $1 - \delta$, we have

- 1) $|\hat{\mu} - \mu| \leq \hat{\epsilon} \leq \epsilon$; &
- 2) $\mathbf{M}^{\star\downarrow} \leq \mu^* \leq \mu \leq \hat{\mu} + \hat{\epsilon} \leq \mathbf{M}^{\star\downarrow} + 2\epsilon$.

5 CONCLUSION

This work generalizes existing theories of fair machine learning, with welfare, malfare, and regret objectives, thus subsuming the *minimax fair learning* [1, 20, 30, 33, 43], *multi-group agnostic PAC learning* [7, 39], and *fair-PAC learning* [14] settings, while enjoying rigorous statistical-learning guarantees and the axiomatization of cardinal welfare theory. In particular, we bound the *generalization error* and *sample complexity* of UCB-optimal models, either given a fixed sample, or to meet a user-supplied ϵ - δ optimality guarantee via progressive sampling. Our bounds leverage the specific character of the objective at hand, and our progressive sampling methods are tailored to three realistic models of data generation. We stress that while training UCB-optimal models is analytically convenient, there is also an important fairness impact to this decision, as fair malfare functions (e.g., egalitarian) place strong emphasis on the most disadvantaged groups, which are often understudied minority groups. Cousins [14] notes that optimizing *empirical malfare* $\hat{\mathbf{M}}$ overfits to small numbers of sampled minorities, however we argue that training UCB-optimal models (i.e., optimizing $\hat{\mathbf{M}}^\uparrow$) factors *uncertainty* into training, so that the needs of understudied groups (i.e., those with large $\hat{\epsilon}_i$ values) are better addressed.

Our active learning setting under the conditional sampling model is philosophically intriguing, as we find that optimally investing sampling effort under uncertainty is challenging, depends on the objective at hand, and has important fairness impact. In §3.3, we see that a host of factors involving the objective, function class \mathcal{H} , and per-group distributions $\mathcal{D}_{1:g}$ all interact to determine the sharpness of welfare, malfare, and regret bounds, and property 3.8 quantifies the incremental UCB improvement of sampling each group. This analysis answers questions raised by Chen et al. [12] as to how sampling-error impacts fairness, and generalizes the analysis of Shekhar et al. [43] from the egalitarian special-case to arbitrary power-mean malfare functions. Algorithm 2 then incorporates these ideas into an *active sampling algorithm*, which dynamically select groups to sample based on projected UCB improvement. Notably, alg. 1 does use uniform sample sizes under the joint sampling model, and uses whatever data are available under the mixture sampling model, as these are natural choices under these sampling models. In contrast, under the conditional sampling model, alg. 2 is able to make more intelligent decisions as to where to allocate sampling effort.

We thus conclude that algorithmic fairness, statistical uncertainty, and sample complexity are tightly intertwined, and must all be considered to best allocate resources in service of the social planner. We are hopeful that this analysis and algorithmic study will lead to a greater emphasis on sample-complexity and finite-sample error analysis for the social planner's problem, which is traditionally analyzed in terms of the asymptotic Bayesian methods of classical economics. In particular, we are hopeful that this analysis emphasizes and mathematically supports the call for greater visibility of minority groups and the importance of incorporating diverse data into (fair) machine learning systems.

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A PROOF APPENDIX

We now present proofs of all mathematical claims in the paper body. We first show in appx. A.1 the estimation guarantees, sample complexity, and incremental-sampling knowledge-gain bounds of §3. We then derive in appx. A.2 the results of §3, i.e., those related to sampling schedules and our progressive sampling algorithms.

A.1 Statistical Estimation Guarantees for Malfare, Welfare, and Regret

Here we show all bounds related to uniform convergence of sentiment values, and estimating welfare, malfare, and regret. We begin with thms. 3.1 and 3.2 and property 3.3, which bound single-group uniform convergence rates in the form of additive error scalar bounds, i.e., $\text{AES}(\dots)$, under various conditions.

Theorem 3.1 (Uniform Convergence for Bounded Finite Hypothesis Classes). We may bound the distribution-free $\text{AES}(m, \delta)$, the distribution-dependent $\text{AES}_{\mathcal{D}}(m, \delta)$, and the data-dependent $\text{AES}(m, \delta, \mathbf{x}, \mathbf{y})$ scalar additive error as

$$1) \varepsilon \leftarrow \sqrt{\frac{r^2 \ln \frac{2|\mathcal{H}|}{\delta}}{2m}} \quad [27]; \quad 2) \varepsilon \leftarrow \frac{r \ln \frac{2|\mathcal{H}|}{\delta}}{3m} + \sup_{h \in \mathcal{H}} \sqrt{\frac{2 \mathbb{V}_{\mathcal{D}}[s \circ h] \ln \frac{2|\mathcal{H}|}{\delta}}{m}} \quad [6]; \quad \& \quad 3) \hat{\varepsilon} \leftarrow \frac{7r \ln \frac{2|\mathcal{H}|+1}{\delta}}{3(m-1)} + \sup_{h \in \mathcal{H}} \sqrt{\frac{2 \hat{\mathbb{V}}_{\mathbf{x}, \mathbf{y}}[s \circ h] \ln \frac{2|\mathcal{H}|+1}{\delta}}{(m-1)}} \quad [16].$$

PROOF. First note that items 1 and 2 hold via union bound over the lower and upper tails of the sentiment of each $h \in \mathcal{H}$, hence the $\ln \frac{2|\mathcal{H}|}{\delta}$ term, and item 3 has a union bound over one additional tail, which is used to bound the supremum variance. In particular, item 1 holds via Hoeffding's [27] inequality, and item 2 via Bennett's inequality [6] (technically the sub-gamma form of Bennett's sub-Poisson bound, a.k.a. Bernstein's inequality), and item 3 holds via the supremum variance bound of Cousins and Riondato [16, thm. 2], followed by Bennett's inequality. Note that here $\hat{\mathbb{V}}[\cdot]$ denotes the unbiased (Bessel-corrected) empirical variance, though the empirical supremum variance is still an upward-biased estimate of the supremum variance. \square

Theorem 3.2 (Uniform Convergence with Rademacher Averages). Suppose hypothesis class \mathcal{H} and sentiment function $s(\cdot, \cdot)$, take $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}^m$ and $\sigma \sim \mathcal{U}^m(\pm 1)$, i.e., σ is uniformly distributed on $(\pm 1)^m$, and define the *Rademacher average* $\mathfrak{R}_m(s \circ \mathcal{H}, \mathcal{D})$ and *Bousquet variance proxy* $\mathfrak{V}_m(s \circ \mathcal{H}, \mathcal{D})$ [see 9] as

$$\mathfrak{R}_m(s \circ \mathcal{H}, \mathcal{D}) \doteq \mathbb{E}_{\mathbf{x}, \mathbf{y}, \sigma} \left[\sup_{h \in \mathcal{H}} \left| \frac{1}{m} \sum_{i=1}^m \sigma_i (s \circ h)(\mathbf{x}_i, \mathbf{y}_i) \right| \right], \quad \mathfrak{V}_m(s \circ \mathcal{H}, \mathcal{D}) \doteq \sup_{h \in \mathcal{H}} \mathbb{V}[s \circ h] + 4r \mathfrak{R}_m(s \circ \mathcal{H}, \mathcal{D}). \quad (7)$$

We may then bound $\text{AES}_{\mathcal{D}}(m, \delta)$ as $\varepsilon \leftarrow 2\mathfrak{R}_m(s \circ \mathcal{H}, \mathcal{D}) + \frac{r \ln \frac{1}{\delta}}{3m} + \sqrt{\frac{2\mathfrak{V}_m(s \circ \mathcal{H}, \mathcal{D}) \ln \frac{1}{\delta}}{m}}$.

PROOF. This result follows via the Rademacher symmetrization inequality [8, lemma 11.4], which upper-bounds the expected supremum deviation as twice the Rademacher average, followed by Bousquet's [2002] bound on the supremum deviation, i.e., on the quantity $\text{AES}_{\mathcal{D}}(\dots)$. \square

Property 3.3 (Practical Bounds on Rademacher Averages). Suppose as in thm. 3.2. The following then hold.

1) Suppose \mathcal{H} has Vapnik-Chervonenkis (VC) dimension d [46, 47], and $\ell(y', y) \doteq 1 - \mathbb{1}_{\{y\}}(y')$ is the 0-1 loss. Then for some absolute constant c , $\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) \leq \sqrt{\frac{cd}{m}}$, which implies bounds for linear classifiers, bounded-depth decision trees [31], and many classes of neural network [2].

2) Suppose $\mathcal{X} \doteq \{\vec{x} \in \mathbb{R}^\infty \mid \|\vec{x}\|_2 \leq R\}$ is the R -radius \mathcal{L}_2 ball in \mathbb{R}^∞ , $\mathcal{H} \doteq \{\vec{x} \mapsto \vec{w} \cdot \vec{x} \mid \|\vec{w}\|_2 \leq \gamma\}$ is a γ -regularized linear class, $\mathcal{Y} \doteq [-R\gamma, R\gamma]$, and $\ell(\cdot, \cdot)$ is a λ -Lipschitz loss function s.t. $\ell(y, y) = 0$. Then $r \leq 2\lambda R\gamma$ and $\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) \leq \frac{2\lambda R\gamma}{\sqrt{m}}$. This implies bounds for (kernelized) SVM, generalized linear models [35], and bounded linear regression.

PROOF. This result is a collection of standard results in statistical learning theory, most of which are fully cited in the property statement. We now carry out our bibliographic duties with regards to the remaining results. These bounds

are often stated for the *empirical Rademacher average*, $\hat{\mathfrak{R}}_m(\ell \circ \mathcal{H}, (\mathbf{x}, \mathbf{y}))$ which conditions on the sample (\mathbf{x}, \mathbf{y}) , but we state bounds that hold over *any possible distribution* by considering *worst-case* realizations of these samples.

We first show item 1. The bound $\mathfrak{R}_m(\mathcal{H}, \mathcal{D}) \leq \sqrt{\frac{cd}{m}}$ follows via Dudley's chaining or entropy integral arguments, along with Haussler's bound on the \mathcal{L}_2 covering number of Vapnik-Chervonenkis classes, see, e.g., Boucheron et al. [8, thm. 13.7]. The specific classes mentioned in the theorem statement are of bounded VC dimension, hence their inclusion.

We now show item 2. First note that each of the cited model classes is Lipschitz-continuous, so their bounds follow from the general statement. Now, note that the range of \mathcal{H} is $[-R\gamma, R\gamma]$, which follows directly from the Cauchy-Schwarz inequality. Composition with $\ell(\cdot, \mathbf{y})$, for $\mathbf{y} \in [-R\gamma, R\gamma]$, then maps this range to a subset of $[0, 2\lambda R\gamma]$, by Lipschitz continuity and nonnegativity of $\ell(\cdot, \cdot)$, and the fact that $\ell(\mathbf{y}, \mathbf{y}) = 0$, hence we conclude $r \leq 2\lambda R\gamma$.

We now bound the Rademacher average. In particular, Shalev-Shwartz and Ben-David [42, lemma 26.10] show that $\mathfrak{R}_m(\mathcal{H}, \mathcal{D}_{|\mathcal{X}}) \leq \frac{R\gamma}{\sqrt{m}}$, where $\mathcal{D}_{|\mathcal{X}}$ denotes the marginalization of the label space \mathcal{Y} from the instance distribution \mathcal{D} , i.e., they bound the Rademacher average of the linear hypothesis class on the unlabeled distribution over \mathcal{X} . Technically, their definition of the Rademacher average contains no absolute value inside the supremum, but this is immaterial, as Cousins and Riondato [16, lemma 5] show that the same bound holds with the absolute value. We then apply the Ledoux-Talagrand contraction principle⁵ [8, lemma 11.6] to compose the hypothesis class with a λ -Lipschitz loss function, which yields $\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) \leq 2\lambda \mathfrak{R}_m(\mathcal{H}, \mathcal{D}_{|\mathcal{X}}) \leq \frac{2\lambda R\gamma}{\sqrt{m}}$. \square

We now show thms. 3.4 and 3.5 and cor. 3.6, which bound the tails and expectations of malfare and welfare values in terms of additive error vector bounds, i.e., in terms of $\text{AEV}(\dots)$.

Theorem 3.4 (Welfare and Malfare Tail Bounds). Suppose sentiment function $s(\cdot, \cdot) : \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$, per-group probability distributions $\mathcal{D}_{1:g}$, sample size vector $\mathbf{m} \in \mathbb{Z}_+^g$, samples $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}_1^{\mathbf{m}_1} \times \dots \times \mathcal{D}_g^{\mathbf{m}_g}$, failure probability $\delta \in (0, 1)$, and additive error bound $\text{AEV}(\dots)$, and let $\hat{\boldsymbol{\varepsilon}} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$. Then for all $h \in \mathcal{H}$ and all monotonic aggregator functions $M(\cdot; \mathbf{w})$, it holds with probability at least $1 - \delta$ over \mathbf{x}, \mathbf{y} , and $\hat{\boldsymbol{\varepsilon}}$ that

$$\underbrace{M\left(i \mapsto 0 \vee \mathbb{E}_{\mathcal{D}_i} [s \circ h] - \hat{\boldsymbol{\varepsilon}}_i; \mathbf{w}\right)}_{\text{TRUE LB}} \leq \underbrace{M\left(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h]; \mathbf{w}\right)}_{\text{PLUG-IN ESTIMATE } \hat{M}} \leq \underbrace{M\left(i \mapsto c \wedge \mathbb{E}_{\mathcal{D}_i} [s \circ h] + \hat{\boldsymbol{\varepsilon}}_i; \mathbf{w}\right)}_{\text{TRUE UB}}, \quad \& \quad (9)$$

$$\underbrace{M\left(i \mapsto 0 \vee \hat{\mathbb{E}}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h] - \hat{\boldsymbol{\varepsilon}}_i; \mathbf{w}\right)}_{\text{LCB ESTIMATE } \hat{M}^\dagger} \leq \underbrace{M\left(i \mapsto \mathbb{E}_{\mathcal{D}_i} [s \circ h]; \mathbf{w}\right)}_{\text{TRUE AGGREGATE } M} \leq \underbrace{M\left(i \mapsto c \wedge \hat{\mathbb{E}}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h] + \hat{\boldsymbol{\varepsilon}}_i; \mathbf{w}\right)}_{\text{UCB ESTIMATE } \hat{M}^\uparrow}, \quad (10)$$

thus if $M(\cdot; \mathbf{w})$ is λ -Lipschitz-continuous w.r.t. some norm $\|\cdot\|_M$, we have

$$\left| \underbrace{M\left(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}} [s \circ h]; \mathbf{w}\right)}_{\text{PLUG-IN ESTIMATE}} - \underbrace{M\left(i \mapsto \mathbb{E}_{\mathcal{D}_i} [s \circ h]; \mathbf{w}\right)}_{\text{TRUE AGGREGATE}} \right| \leq \lambda \|\hat{\boldsymbol{\varepsilon}}\|_M. \quad (11)$$

PROOF. We first note that, by the definition of $\text{AEV}(\dots)$, with probability at least $1 - \delta$, for all $h \in \mathcal{H}$ and groups $i \in \mathcal{Z}$, it holds that $|\mathbb{E}_{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^*| \leq \hat{\boldsymbol{\varepsilon}}_i$. Given this, note that (9) and (10) follow directly from the *monotonicity assumption* on $M(\cdot; \mathbf{w})$. Similarly, (11) follows from the monotonicity assumption and the definition of Lipschitz-continuity. \square

⁵It is of course possible to remove the leading 2 factor, if we are willing to work with the Rademacher average definition without the absolute value.

Theorem 3.5 (Welfare and Malfare Expectation Bounds). Suppose as in thm. 3.4, and assume also that $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) = \text{AEV}(\mathbf{m}, \delta)$ is a *distribution-free* or *distribution-dependent* (but *not data-dependent*) bound. Then

$$|M - \mathbb{E}[\hat{M}]| \leq \mathbb{E}[|M - \hat{M}|] \leq \lambda \int_0^1 \|\text{AEV}(\mathbf{m}, \delta)\|_{\mathbb{M}} d\delta .$$

PROOF. First note that by Jensen's inequality and convexity of $|\cdot|$, we have

$$|M - \mathbb{E}[\hat{M}]| \leq \mathbb{E}[|M - \hat{M}|] ,$$

thus all that remains to be shown is

$$\mathbb{E}[|M - \hat{M}|] \leq \lambda \int_0^1 \|\text{AEV}(\mathbf{m}, \delta)\|_{\mathbb{M}} d\delta .$$

By the definition of Lipschitz-continuity, note that for any \mathbf{s}, \mathbf{s}' , it holds that

$$|M(\mathbf{s}; \mathbf{w}) - M(\mathbf{s}'; \mathbf{w})| \leq \lambda \|\mathbf{s} - \mathbf{s}'\|_{\mathbb{M}} .$$

Now, we conclude the desideratum as

$$\begin{aligned} \mathbb{E}[|M - \hat{M}|] &= \int_0^\infty \mathbb{P}(|M - \hat{M}| > \varepsilon) d\varepsilon && \text{PROPERTIES OF EXPECTATION (HAZARD FORMULA)} \\ &\leq \int_0^\infty \mathbb{P}(\lambda \|\mathbf{s} - \hat{\mathbf{s}}\|_{\mathbb{M}} > \varepsilon) d\varepsilon && |M - \hat{M}| \leq \lambda \|\mathbf{s} - \hat{\mathbf{s}}\|_{\mathbb{M}} \\ &= \int_0^1 \inf\{\varepsilon \geq 0 \mid \mathbb{P}(\lambda \|\mathbf{s} - \hat{\mathbf{s}}\|_{\mathbb{M}} > \varepsilon) \leq \delta\} d\delta && \text{INTEGRAL OF INVERSE FORMULA} \\ &\leq \lambda \int_0^1 \|\text{AEV}(\mathbf{m}, \delta)\|_{\mathbb{M}} d\delta . && \text{DEFINITION OF AEV}(\dots) \quad \square \end{aligned}$$

Corollary 3.6 (Bernstein-Type Malfare Bounds). Suppose as in thm. 3.1, and also per-group sample size m (i.e., $\mathbf{m} = \langle m, \dots, m \rangle$) and $p \geq 1$ power-mean malfare function $\mathbb{M}_p(\cdot; \mathbf{w})$. Now, let $\mathbf{v}_i \doteq \mathbb{E}_{\mathcal{D}_i}[s \circ h]$, and define variance proxy v as $v \doteq (\sum_{i \in \mathcal{Z}} \mathbf{w}_i \sqrt{v_i})^2$ for $p = 1$, $v \doteq \mathbf{w} \cdot \mathbf{v}$ for $p \in (1, 2]$, or $v \doteq \|\mathbf{v}\|_\infty$ for $p > 2$. Then, for all $\delta \in (0, 1)$, we have
 1) $\mathbb{P}\left(|\mathbb{M} - \hat{\mathbb{M}}| \geq \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2v \ln \frac{2g}{\delta}}{m}}\right) \leq \delta$; 2) $\mathbb{E}[|\mathbb{M} - \hat{\mathbb{M}}|] \leq \frac{r \ln(2eg)}{3m} + \sqrt{\frac{2v \ln(2eg)}{m}}$; & 3) $\mathbb{M} \leq \mathbb{E}[\hat{\mathbb{M}}] \leq \mathbb{M} + \frac{r \ln(eg)}{3m} + \sqrt{\frac{2v \ln(eg)}{m}}$.

PROOF. We first show the tail bounds (item 1), which we then use to show the expectation bounds (items 2 and 3). In particular, we apply thm. 3.1 item 2 to the singleton function family $\mathcal{H} \doteq \{h\}$ (thus $|\mathcal{H}| = 1$), with a union bound over all g groups to bound per-group confidence radii, i.e.,

$$\varepsilon_i \leftarrow \frac{r \ln \frac{2g}{\delta}}{3m} + \sup_{h \in \mathcal{H}} \sqrt{\frac{2 \mathbb{V}_{\mathcal{D}_i}[s \circ h] \ln \frac{2g}{\delta}}{m}} = \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2v_i \ln \frac{2g}{\delta}}{m}} .$$

We then apply (11) of thm. 3.4 to bound malfare tails in terms of these confidence radii, via the Lipschitz property of power-mean malfare functions, i.e., thm. 2.2 item 3. Subsequently, thm. 3.5 converts these tail bounds into expectation bounds for item 2, and similar logic for 1-tailed bounds yields item 3. In all cases, $\mathbb{M}_p(\varepsilon; \mathbf{w})$ (and integrals thereof) are the key quantities that we must bound. The resulting bounds have rather convenient forms for $p \in \{1, 2, \infty\}$, so we relate all other cases to these three via monotonicity.

We first show the case of $p = 1$. Observe that

$$\begin{aligned} \Lambda_1(\varepsilon; \mathbf{w}) &= \sum_{i \in \mathcal{Z}} \mathbf{w}_i \left(\frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2v_i \ln \frac{2g}{\delta}}{m}} \right) && \text{DEFINITION OF } \Lambda_1(\cdot; \mathbf{w}), \varepsilon \\ &= \frac{r \ln \frac{2g}{\delta}}{3m} + \left(\sum_{i \in \mathcal{Z}} \mathbf{w}_i \sqrt{v_i} \right) \sqrt{\frac{2 \ln \frac{2g}{\delta}}{m}} = \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2 \left(\sum_{i \in \mathcal{Z}} \mathbf{w}_i \sqrt{v_i} \right)^2 \ln \frac{2g}{\delta}}{m}}. && \text{ALGEBRA} \end{aligned}$$

Now suppose $p \in [1, 2]$. Then

$$\begin{aligned} \Lambda_p(\varepsilon; \mathbf{w}) &\leq \Lambda_2(\varepsilon; \mathbf{w}) && \text{POWER-MEAN INEQUALITY} \\ &= \sqrt{\sum_{i \in \mathcal{Z}} \mathbf{w}_i \left(\frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2v_i \ln \frac{2g}{\delta}}{m}} \right)^2} && \text{DEFINITION OF } \Lambda_2(\cdot; \mathbf{w}), \varepsilon \\ &= \sqrt{\left(\frac{r \ln \frac{2g}{\delta}}{3m} \right)^2 + 2 \frac{r \ln \frac{2g}{\delta}}{m} \cdot \frac{\sum_{i \in \mathcal{Z}} \mathbf{w}_i \sqrt{2v_i \ln \frac{2g}{\delta}}}{\sqrt{m}} + \frac{2\mathbf{w} \cdot v \ln \frac{2g}{\delta}}{m}} && \text{ALGEBRA} \\ &\leq \sqrt{\left(\frac{r \ln \frac{2g}{\delta}}{3m} \right)^2 + 2 \frac{r \ln \frac{2g}{\delta}}{m} \sqrt{\frac{2 \sum_{i \in \mathcal{Z}} \mathbf{w}_i v_i \ln \frac{2g}{\delta}}{m}} + \frac{2\mathbf{w} \cdot v \ln \frac{2g}{\delta}}{m}} && \text{JENSEN'S INEQUALITY} \\ &= \sqrt{\left(\frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2\mathbf{w} \cdot v \ln \frac{2g}{\delta}}{m}} \right)^2} = \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2\mathbf{w} \cdot v \ln \frac{2g}{\delta}}{m}}. && \text{FACTORING} \end{aligned}$$

Finally, for $p \in [1, \infty]$, we have

$$\Lambda_p(\varepsilon; \mathbf{w}) \leq \Lambda_\infty(\varepsilon; \mathbf{w}) = \max_{i \in \mathcal{Z}} \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2v_i \ln \frac{2g}{\delta}}{m}} = \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2\|\mathbf{v}\|_\infty \ln \frac{2g}{\delta}}{m}}.$$

This yields the final case of the result. In all three cases, the tail bounds of item 1 then follows by substitution into (11).

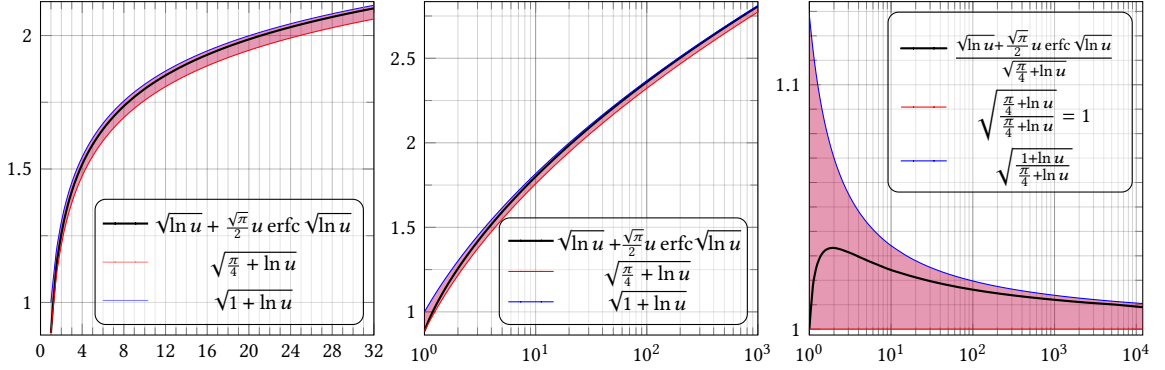
We now show the *expectation bounds* of items 2 and 3. Via thm. 3.5, we seek to *bound integrals* of the form

$$\mathbb{E}[|\Lambda - \hat{\Lambda}|] \leq \int_0^1 \Lambda_p(\text{AEV}(\langle m, \dots, m \rangle, \delta); \mathbf{w}) \, d\delta \leq \int_0^1 \sqrt{\frac{2v \ln \frac{g}{\delta}}{m}} + \frac{r \ln \frac{g}{\delta}}{3m} \, d\delta,$$

for the various v as defined above. Note that we will encounter either $\ln \frac{2g}{\delta}$ or $\ln \frac{g}{\delta}$ terms, depending on whether we employ 2-tailed or 1-tailed bounds, in items 2 or 3, respectively. The difficult part of these integrals is

$$\begin{aligned} \int_0^1 \sqrt{\ln \frac{g}{\delta}} \, d\delta &= \frac{1}{2} \sqrt{\pi g} \operatorname{erfc}(\sqrt{\ln g}) + \sqrt{\ln g} && \text{INTEGRATION} \\ &\leq \sqrt{\ln(eg)}. && \forall u \geq 1: \sqrt{\ln u} + u^{\frac{1}{2}} \sqrt{\pi} \operatorname{erfc}(\sqrt{\ln u}) \leq \sqrt{\ln(eu)} \end{aligned}$$

Note that the integration is exact, and the looseness of the $\operatorname{erfc}(\cdot)$ (complementary error function) bound is quite minuscule, both multiplicatively and additively, as $\forall u \geq 1: 0 \leq \sqrt{\ln(eu)} - (\sqrt{\ln u} + u^{\frac{1}{2}} \sqrt{\pi} \operatorname{erfc}(\sqrt{\ln u})) \leq 1 - \frac{\sqrt{\pi}}{2}$, see fig. A1 for visualization. Observe also that the logarithmic terms in the fast-decaying summands integrate (improperly)



(a) Natural axes plot. Note that the shaded region between the lower and upper bounds is difficult to see without scaling, and its height is maximized at $u = 1$. (b) Semilog plot to emphasize asymptotic behavior. Note that even with a logarithmic x -axis, the absolute gap between the lower and upper bounds remains quite small. (c) Semilog plot, wherein each function is divided by the lower bound $\sqrt{\frac{\pi}{4} + \ln u}$ to emphasize the decaying relative gap between lower and upper bounds.

Fig. A1. Plots of various scalings of $\sqrt{\ln u} + \frac{\sqrt{\pi}}{2} u \operatorname{erfc} \sqrt{\ln u}$ and lower and upper bounds thereof (y -axis), i.e., $\sqrt{\frac{\pi}{4} + \ln u} \leq \sqrt{\ln u} + \frac{\sqrt{\pi}}{2} u \operatorname{erfc} \sqrt{\ln u} \leq \sqrt{1 + \ln u}$, see proof of cor. 3.6, against $u \geq 1$ (x -axis). The region sandwiched by the lower and upper bounds is shaded in purple, is quite small for all $u \geq 1$, and converges to 0 both additively and multiplicatively.

as

$$\int_0^1 \ln \frac{g}{\delta} d\delta = \ln(g) - \int_0^1 \ln \delta d\delta = \ln(g) + 1 = \ln(eg) .$$

Substituting the values of v derived above into these indefinite integrals then yields the desiderata. In particular, item 2 substitutes into thm. 3.5 directly to bound $\mathbb{E}[|\mathcal{M} - \hat{\mathcal{M}}|]$, and item 3 substitutes into a directed variant of thm. 3.5 (i.e., one considering single-sided error, rather than absolute error) to upper-bound $\mathbb{E}[\hat{\mathcal{M}}]$, paired with the observation that, by Jensen's inequality (as $p \geq 1$ power means are convex), $\mathcal{M} \leq \mathbb{E}[\hat{\mathcal{M}}]$. \square

We now show thm. 3.7, which provides regret bounds analogous to the aggregator function bounds of thm. 3.4.

Theorem 3.7 (Regret Estimation Bounds). Suppose sentiment function $s(\cdot, \cdot): \mathcal{Y}' \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$, per-group probability distributions $\mathcal{D}_{1:g}$, sample size vector $\mathbf{m} \in \mathbb{Z}_+^g$, samples $(\mathbf{x}, \mathbf{y}) \sim \mathcal{D}_1^{\mathbf{m}_1} \times \dots \times \mathcal{D}_g^{\mathbf{m}_g}$, failure probability $\delta \in (0, 1)$, and additive error bound $\text{AEV}(\dots)$, and let $\hat{\varepsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$. Then for all $h \in \mathcal{H}$ and all monotonic malfare functions $\mathcal{M}(\cdot; \mathbf{w})$, it holds with probability at least $1 - \delta$ over \mathbf{x}, \mathbf{y} , and $\hat{\varepsilon}$ that

$$\underbrace{\mathcal{M}\left(i \mapsto 0 \vee \left| \frac{\mathbb{E}_{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^*}{\mathcal{D}_i} \right| - 2\hat{\varepsilon}_i; \mathbf{w}\right)}_{\text{TRUE REGRET MALFARE LB}} \leq \underbrace{\mathcal{M}\left(i \mapsto \left| \frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i}{\mathbf{x}_i, \mathbf{y}_i} \right|; \mathbf{w}\right)}_{\text{PLUG-IN REGRET MALFARE}} \leq \underbrace{\mathcal{M}\left(i \mapsto c \wedge \left| \frac{\mathbb{E}_{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^*}{\mathcal{D}_i} \right| + 2\hat{\varepsilon}_i; \mathbf{w}\right)}_{\text{TRUE REGRET MALFARE UB}}, \quad \& \quad (14)$$

$$\underbrace{\mathcal{M}\left(i \mapsto 0 \vee \left| \frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i}{\mathbf{x}_i, \mathbf{y}_i} \right| - 2\hat{\varepsilon}_i; \mathbf{w}\right)}_{\text{LCB ESTIMATE}} \leq \underbrace{\mathcal{M}\left(i \mapsto \left| \frac{\mathbb{E}_{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^*}{\mathcal{D}_i} \right|; \mathbf{w}\right)}_{\text{TRUE REGRET MALFARE}} \leq \underbrace{\mathcal{M}\left(i \mapsto c \wedge \left| \frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i}{\mathbf{x}_i, \mathbf{y}_i} \right| + 2\hat{\varepsilon}_i; \mathbf{w}\right)}_{\text{UCB ESTIMATE}}, \quad (15)$$

thus if $\mathcal{M}(\cdot; \mathbf{w})$ is λ -Lipschitz-continuous w.r.t. some norm $\|\cdot\|_{\mathcal{M}}$, we have

$$\underbrace{\mathcal{M}\left(i \mapsto \left| \frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i}{\mathbf{x}_i, \mathbf{y}_i} \right|; \mathbf{w}\right)}_{\text{PLUG-IN REGRET MALFARE}} - \underbrace{\mathcal{M}\left(i \mapsto \left| \frac{\mathbb{E}_{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^*}{\mathcal{D}_i} \right|; \mathbf{w}\right)}_{\text{TRUE REGRET MALFARE}} \leq 2\lambda \|\hat{\varepsilon}\|_{\mathcal{M}} . \quad (16)$$

PROOF. This result follows essentially the same reasoning as thm. 3.4. The salient difference here is that we now bound both the estimation error of h and each \mathbf{h}_i^* , which when summed yield the novel 2-factors. In particular, we have that with probability at least $1 - \delta$, it holds for each $i \in \mathcal{Z}$ and $h \in \mathcal{H}$ (simultaneously) that

$$\begin{aligned} \left| \frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^* \right| &= \left| \left(\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i \right) + \left(\frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - \mathbf{s}_i^* \right) - \left(\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i \right) \right| && \text{ALGEBRA} \\ &\leq \left| \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i \right| + \left| \frac{\mathbb{E}}{\mathcal{D}_i} [s \circ h] - \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] \right| + \left| \mathbf{s}_i^* - \hat{\mathbf{s}}_i \right| && \text{TRIANGLE INEQUALITY} \\ &\leq \left| \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i \right| + \hat{\varepsilon}_i + \hat{\varepsilon}_i = \left| \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [s \circ h] - \hat{\mathbf{s}}_i \right| + 2\hat{\varepsilon}_i . && \text{DEFINITION OF } \hat{\varepsilon} \end{aligned}$$

This result, paired with the assumed monotonicity of $\mathbb{M}(\cdot; \mathbf{w})$, yields (14) and (15), and then applying the definition of Lipschitz continuity yields (16). \square

We now show property 3.8, which approximates the reduction to the UCB resultant from drawing a single additional sample for some group i .

Property 3.8 (Incremental Gain of Sampling). Suppose power-mean malfare $\mathbb{M}_p(\cdot; \mathbf{w})$, sample (\mathbf{x}, \mathbf{y}) with group sample sizes $\mathbf{m}_{1:q}$, and let \mathbf{x}', \mathbf{y}' extend \mathbf{x}, \mathbf{y} to sample sizes \mathbf{m}' , where $\mathbf{m}' = \mathbf{m} + \mathbb{1}_i$, i.e., group i has one additional sample. Now, let $\hat{\varepsilon} \leftarrow \text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y})$ and $\tilde{\varepsilon} \leftarrow \text{AEV}(\mathbf{m}', \delta, \mathbf{x}', \mathbf{y}')$, and take $\hat{h} \doteq \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{M}_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ h] + \hat{\varepsilon}_i; \mathbf{w})$, $\hat{\mathbb{M}} \doteq \mathbb{M}_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}]; \mathbf{w})$, $\hat{\mathbb{M}}^\dagger \doteq \mathbb{M}_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\varepsilon}_i; \mathbf{w})$, and $\tilde{\mathbb{M}}^\dagger \doteq \inf_{h \in \mathcal{H}} \mathbb{M}_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}'_i, \mathbf{y}'_i} [\ell \circ h] + \tilde{\varepsilon}_i; \mathbf{w})$.

Then the *incremental impact* of sampling from group i on the UCB is approximately

$$\hat{\mathbb{M}}^\dagger - \tilde{\mathbb{M}}^\dagger \approx \frac{\hat{\varepsilon}_i \mathbf{w}_i}{2\mathbf{m}_i + \frac{3}{2}} \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\varepsilon}_i}{\hat{\mathbb{M}}^\dagger} \right)^{p-1} \approx \frac{\hat{\varepsilon}_i \mathbf{w}_i}{2\mathbf{m}_i} \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}]}{\hat{\mathbb{M}}} \right)^{p-1}. \quad (17)$$

PROOF. We first assume that for each $h \in \mathcal{H}$, $i \in \mathcal{Z}$, it holds that $\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ h] \approx \hat{\mathbb{E}}_{\mathbf{x}'_i, \mathbf{y}'_i} [\ell \circ h]$, and we use this approximation throughout. The result then follows directly from three observations.

First, note that $\frac{\partial}{\partial \mathbf{s}_i} \mathbb{M}_p(\mathbf{s}; \mathbf{w}) = \frac{\mathbf{w}_i s_i^{p-1}}{\mathbb{M}_p^{p-1}(\mathbf{s}; \mathbf{w})}$, thus for any UCB-optimal \hat{h} , we have the *subderivative*⁶

$$\mathbf{w}_i \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\varepsilon}_i}{\hat{\mathbb{M}}^\dagger} \right)^{p-1} \in \frac{\partial_{\text{sub}}}{\partial_{\text{sub}} \hat{\varepsilon}_i} \inf_{h \in \mathcal{H}} \mathbb{M}_p \left(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\varepsilon}_i; \mathbf{w} \right).$$

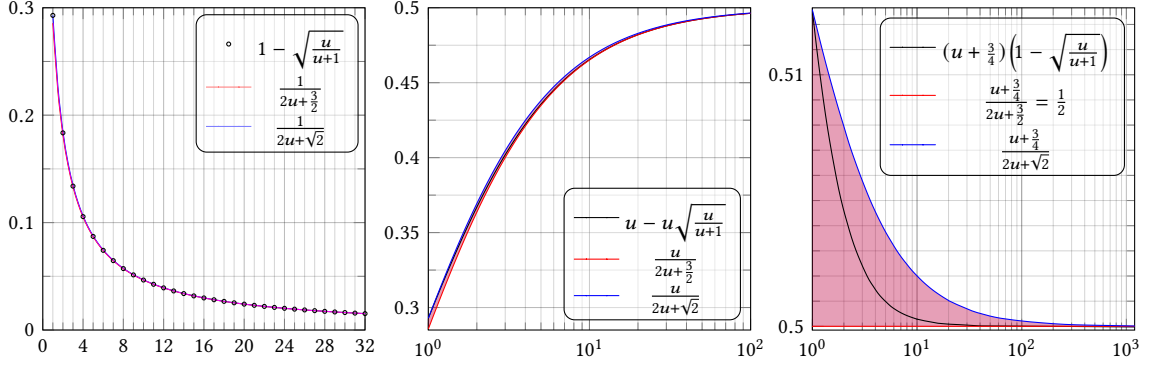
Now, assuming $\Theta \sqrt{\frac{1}{\mathbf{m}_i}}$ asymptotic behavior of $\hat{\varepsilon}_i$, observe that

$$\tilde{\varepsilon}_i \approx \hat{\varepsilon}_i \sqrt{\frac{\mathbf{m}_i}{\mathbf{m}_i + 1}} \implies (\hat{\varepsilon}_i - \tilde{\varepsilon}_i) \approx \hat{\varepsilon}_i \left(1 - \sqrt{\frac{\mathbf{m}_i}{\mathbf{m}_i + 1}} \right) \approx \frac{\hat{\varepsilon}_i}{2\mathbf{m}_i + \frac{3}{2}} \approx \frac{\hat{\varepsilon}_i}{2\mathbf{m}_i},$$

and note that this approximation is quite sharp (see fig. A2), as

$$\forall u \geq 1 : \frac{1}{2u + \frac{3}{2}} \leq 1 - \sqrt{\frac{u}{u+1}} \leq \frac{1}{2u + \sqrt{2}}. \quad (21)$$

⁶Technically, $\inf_{h \in \mathcal{H}} \mathbb{M}_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\varepsilon}_i; \mathbf{w})$ is not always convex in $\hat{\varepsilon}$, thus this quantity is not always a subderivative, but it is at least a reasonable linear approximation. Furthermore, a sufficient condition for convexity and the stated subderivative to hold is for $\mathbb{M}_p(i \mapsto \hat{\mathbb{E}}_{\mathbf{x}_i, \mathbf{y}_i} [\ell \circ \hat{h}] + \hat{\varepsilon}_i; \mathbf{w})$, which is always convex in $\hat{\varepsilon}$, to also be *jointly convex* in $\hat{\varepsilon}$ and in the space of $\hat{h} \in \mathcal{H}$ (as is quite common in machine learning and optimization contexts).



(a) Natural axes plot. Note that the shaded region between the lower and upper bounds is difficult to see without scaling. (b) Semilog plot, wherein each function is scaled by u to emphasize their asymptotic $\Theta(\frac{1}{u})$ behavior. (c) Semilog plot, wherein each function is scaled by $(u + \frac{3}{4})$ to better illustrate the gap between lower and upper bounds.

Fig. A2. Plots of various scalings of $1 - \sqrt{\frac{u}{u+1}}$ and lower and upper bounds thereof (y -axis), i.e., $\frac{1}{2u+\frac{3}{2}} \leq 1 - \sqrt{\frac{u}{u+1}} \leq \frac{1}{2u+\sqrt{2}}$, see (21), against $u \geq 1$ (x -axis). The region sandwiched by the lower and upper bounds is shaded in purple, and we note that even when scaled by u , this region rapidly converges to 0 — in particular, the gap is bounded as $\frac{1}{2u+\sqrt{2}} - \frac{1}{2u+\frac{3}{2}} \leq \frac{\frac{3}{2}-\sqrt{2}}{4u^2}$.

Finally, using the subderivative approximation, the impact of the additional sample on the malfare is approximately

$$\begin{aligned}
\hat{M}^\dagger - \tilde{M}^\dagger &\approx (\hat{\varepsilon}_i - \tilde{\varepsilon}_i) \frac{\partial_{\text{sub}}}{\partial_{\text{sub}} \hat{\varepsilon}_i} \inf_{h \in \mathcal{H}} \mathbb{M}_p \left(j \mapsto \hat{\mathbb{E}}_{\mathbf{x}_{j::}, \mathbf{y}_{j::}} [\ell \circ \hat{h}] + \hat{\varepsilon}_j; \mathbf{w} \right) && \text{FINITE DIFFERENCE APPROXIMATION} \\
&\approx \hat{\varepsilon}_i \left(1 - \sqrt{\frac{m_i}{m_i + 1}} \right) \mathbf{w}_i \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_{i::}, \mathbf{y}_{i::}} [\ell \circ \hat{h}] + \hat{\varepsilon}_i}{\hat{M}^\dagger} \right)^{p-1} && \hat{\mathbb{E}}_{\mathbf{x}_{i::}, \mathbf{y}_{i::}} [\ell \circ h] \approx \hat{\mathbb{E}}_{\mathbf{x}_{i::}, \mathbf{y}_{i::}} [\ell \circ h] \\
&\approx \frac{\hat{\varepsilon}_i}{2m_i + \frac{3}{2}} \mathbf{w}_i \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_{i::}, \mathbf{y}_{i::}} [\ell \circ \hat{h}] + \hat{\varepsilon}_i}{\hat{M}^\dagger} \right)^{p-1} && \text{SEE ABOVE} \\
&\approx \frac{\hat{\varepsilon}_i \mathbf{w}_i}{2m_i} \left(\frac{\hat{\mathbb{E}}_{\mathbf{x}_{i::}, \mathbf{y}_{i::}} [\ell \circ \hat{h}]}{\hat{M}} \right)^{p-1} && \text{SEE ABOVE}
\end{aligned}$$

$$\frac{1}{m_i + \Theta(1)} \approx \frac{1}{m_i}$$

$$\frac{s_i + \varepsilon_i}{\mathbb{M}_p(s + \varepsilon; \mathbf{w})} \approx \frac{s_i}{\mathbb{M}_p(s; \mathbf{w})} \quad \square$$

A.2 Uniformly-Convergent Sampling Schedules and Progressive Sampling Guarantees

In this subsection, we show all results relating to sampling schedules and progressive sampling. Before showing the correctness of algs. 1 and 2, we begin with the uniformly-convergent schedule analysis of lemma 4.4.

Lemma 4.4 (Sufficient Conditions for Uniformly-Convergent Geometric Schedules). Suppose as in def. 4.2, and assume

$$\sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{S_T \times g}} \|\text{AEV}(\langle S_T, \dots, S_T \rangle, \frac{\delta}{T}, \mathbf{x}, \mathbf{y})\|_{\mathbb{M}} \leq \varepsilon. \quad (19)$$

Then the geometric-uniform schedule (S, Δ) is ε -uniformly-convergent. Furthermore, suppose as in def. 4.3, $\alpha \geq \frac{1}{8}$ and

$$\lim_{m \rightarrow \infty} \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{m \times g}} \|\text{AEV}(\langle m, \dots, m \rangle, \frac{\beta-1}{\beta(m+1)}, \mathbf{x}, \mathbf{y})\|_{\mathbb{M}} = 0. \quad (20)$$

Then the double-geometric schedule (S, Δ) is 0-uniformly-convergent.

PROOF. Recall that, via (18) of def. 4.1, the goal is to show that

$$\inf_{t \in \mathbb{Z}_+} \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{S_t \times g}} \|\text{AEV}(\langle S_t, \dots, S_t \rangle, \Delta_t, \mathbf{x}, \mathbf{y})\|_{\mathbb{M}} \leq \varepsilon .$$

We now show the results for the geometric-uniform and double-geometric schedules in separate parts.

The result for the geometric-uniform schedule, i.e., (19), holds by the regularity conditions assumed on $\text{AEV}(\dots)$, where the infimum occurs, possibly nonuniquely, at $t = T$. In particular, this is implied by condition 3, thus we have

$$\begin{aligned} \inf_{t \in \mathbb{Z}_+} \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{S_t \times g}} \|\text{AEV}(\langle S_t, \dots, S_t \rangle, \Delta_t, \mathbf{x}, \mathbf{y})\|_{\mathbb{M}} &= \sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{S_T \times g}} \left\| \text{AEV}(\langle S_T, \dots, S_T \rangle, \frac{\delta}{T}, \mathbf{x}, \mathbf{y}) \right\|_{\mathbb{M}} && \text{CONDITION 3} \\ &\leq \varepsilon . && \text{BY ASSUMPTION} \end{aligned}$$

We may thus conclude that the geometric-uniform schedule is ε -uniformly-convergent.

The result for the double-geometric schedule, i.e., (20), uses the regularity conditions on our bound $\text{AEV}(\dots)$, and also derives a bound on Δ_t in terms of S_t , which allows us to reason over only sample sizes, rather than the schedule itself. Let $t(m)$ denote the smallest timestep t for which a sample of size m is guaranteed to be drawn. In particular, ignoring the ceiling operator, and the fact that not every possible sample size $m \in \mathbb{Z}_+$ is in the schedule S , the inverse of $m = \alpha \beta^t$ gives us $t(m) \geq \log_{\beta} \frac{m}{\alpha}$. However, the actual sample size m may be up to a factor β larger due to the discretization of the geometric schedule, and it may also be rounded up, thus we have the matching upper-bound $t(m) \leq \log_{\beta} \frac{\beta(m+1)}{\alpha}$. With this result in hand, we now substitute and simplify to get

$$\begin{aligned} \Delta_{t(m)} &\geq \frac{\delta(\beta - 1)}{\beta^{\log_{\beta} \frac{\beta(m+1)}{\alpha}}} && \begin{array}{l} \text{SEE ABOVE} \\ \text{DEFINITION 4.3} \end{array} \\ &= \frac{\alpha \delta (\beta - 1)}{\beta(m+1)} && \text{ALGEBRA} \\ &\geq \frac{\beta - 1}{\beta(m+1)} . && \alpha \geq \frac{1}{\delta} \end{aligned}$$

Now, observe that $\sup_{(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^{m \times g}} \|\text{AEV}(\langle m, \dots, m \rangle, \delta, \mathbf{x}, \mathbf{y})\|_{\mathbb{M}}$ is monotonically decreasing in m and δ by regularity condition 3, and thus, by the specified limit condition, may be taken arbitrarily close to 0 for sufficient values of m at timestep $t(m)$. We thus conclude that the double-geometric schedule is 0-uniformly-convergent. \square

We now show the correctness results for algs. 1 and 2, namely thms. 4.5 and 4.6.

Theorem 4.5 (Linear PS Guarantees). Suppose $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{\star \downarrow}) \leftarrow \text{LINEARPSLOSS}(\mathcal{H}, \ell(\cdot, \cdot), \mathcal{D}, \text{AEV}(\dots), S, \Delta, \varepsilon, M(\cdot; \mathbf{w}), \text{REG})$, $M(\mathbf{s}; \mathbf{w})$ is continuous and monotonic in \mathbf{s} with (possibly infinite) Lipschitz constant $\lambda_{\mathbb{M}}$ w.r.t. $\|\cdot\|_{\mathbb{M}}$, and the schedules (S, Δ) are $\frac{\varepsilon}{\lambda_{\mathbb{M}}(1 + \mathbb{1}_{\text{REG}})}$ -uniformly-convergent w.r.t. $\text{AEV}(\dots)$ and $\|\cdot\|_{\mathbb{M}}$. Now take μ to be the true objective value of \hat{h} and μ^{\star} to be the true objective value of the optimal h^{\star} , i.e., if $\text{REG} = \text{FALSE}$, take $\mu \doteq M(i \mapsto \mathbb{E}_{\mathcal{D}_i}[\ell \circ \hat{h}]; \mathbf{w})$ and $\mu^{\star} \doteq \inf_{h \in \mathcal{H}} M(i \mapsto \mathbb{E}_{\mathcal{D}_i}[\ell \circ h]; \mathbf{w})$, or if $\text{REG} = \text{TRUE}$, take (see equation 3) $\mu \doteq M(i \mapsto \text{Reg}_i(\hat{h}); \mathbf{w})$ and $\mu^{\star} \doteq \inf_{h \in \mathcal{H}} M(i \mapsto \text{Reg}_i(h); \mathbf{w})$. Then, with probability at least $1 - \delta$, the output $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{\star \downarrow})$ obeys

- 1) $|\hat{\mu} - \mu| \leq \hat{\varepsilon} \leq \varepsilon$; & & 2) $M^{\star \downarrow} \leq \mu^{\star} \leq \mu \leq \hat{\mu} + \hat{\varepsilon} \leq M^{\star \downarrow} + 2\varepsilon$.

PROOF. We first show that alg. 1 is guaranteed to eventually terminate⁷ after reaching some timestep \hat{t} . Note that at timestep t , under the joint sampling model, $\text{AEV}(\dots)$ is evaluated with sample sizes $\mathbf{m} = \langle S_t, \dots, S_t \rangle$, and under the

⁷Technically, under the mixture sampling model it is possible that the loop (lines 9–12) runs infinitely, but so long as each group is sampled with nonzero probability, this is a zero probability event.

mixture sampling model, sample sizes are at least that large, i.e., $\mathbf{m} \geq \langle S_t, \dots, S_t \rangle$, which by the monotonicity regularity condition (item 3) on $\text{AEV}(\dots)$, implies that the bounds with larger sample sizes are at least as sharp (in the worst case; for most reasonable bounds, increasing sample sizes yields improvement).

By the definition of $\frac{\varepsilon}{\lambda_M(1+\mathbb{1}_{\text{REG}})}\text{-}\|\cdot\|_M$ -uniform-convergence (def. 4.1), for finite λ , we eventually obtain a sufficiently accurate estimate (i.e., $\hat{\varepsilon}$ is sufficiently close to 0) such that by the Lipschitz guarantees of thm. 3.4 or thm. 3.7, the algorithm will terminate (line 19). Furthermore, for infinite λ , $\lim_{t \rightarrow \infty} \hat{\varepsilon} = 0$, which again implies eventual termination, now under only the continuity assumption. In particular, both $M^{\star\downarrow}$ and \hat{M}^\uparrow are eventually ε -estimated, for a total error of $\leq 2\varepsilon$, which yields termination.

To see the correctness of this result, we first observe that by union bound over $t \in \mathbb{Z}_+$, all tail bounds of $\text{AEV}(\dots)$ hold simultaneously with probability at least $1 - \sum_{i=1}^{\infty} \Delta_i = 1 - \delta$ (recall that δ is defined as such by regularity condition 2). Under the joint sampling model, this union bound is a simple bound over $\text{AEV}(\dots)$ evaluated at sample size vectors $\langle S_1, \dots, S_1 \rangle, \langle S_2, \dots, S_2 \rangle, \dots$, and failure probability values $\Delta_1, \Delta_2, \dots$. At first glance, this strategy seems invalid for the mixture sampling model, as the sample sizes are actually not known *a priori*, since they depend on the groups sampled, however a subtle conditioning argument circumvents this objection. The simple trick we employ is to condition the algorithm on the infinite sequence of $z \in 2^{\mathcal{Z}}$ samples drawn on line 10, and establish the schedule *after performing this conditioning operation*, thus the sample sizes are fixed from the perspective of the schedule, and the samples themselves remain *conditionally i.i.d.* given this order. This technique is entirely valid, because the sample sizes depend only on these z values, and are *conditionally independent* from the actual samples \mathbf{x} and \mathbf{y} .

We henceforth assume that we are in the probability at least $1 - \delta$ case where all $\hat{\varepsilon}$ bounds taken over the course of the schedule hold. With this established, items 1 and 2 follow via thm. 3.4 if $\text{REG} = \text{TRUE}$, and via thm. 3.7 otherwise. In particular, item 1 holds, as $\hat{h} \in \mathcal{H}$, and $\hat{\varepsilon}$ was computed with $\text{AEV}(\dots)$, we get $|\hat{\mu} - \mu| \leq \hat{\varepsilon}$, and by the termination condition, $\hat{\varepsilon} \leq \varepsilon$. To see item 2, observe that it is implied by item 1, coupled with the bound $M^{\star\downarrow} \leq \mu^*$, which holds for similar reasons. \square

Theorem 4.6 (Braided PS Guarantees). Suppose $(\hat{h}, \hat{\mu}, \hat{\varepsilon}, M^{\star\downarrow}) \leftarrow \text{BRAIDEDPSLOSS}(\mathcal{H}, \ell(\cdot, \cdot), \mathcal{D}, \text{AES}(\dots), S, \Delta, \varepsilon, M(\cdot; \mathbf{w}), \text{REG})$, $M(\mathbf{s}; \mathbf{w})$ is continuous and strictly monotonic in \mathbf{s} with (possibly infinite) Lipschitz constant λ_M w.r.t. $\|\cdot\|_M$, and the schedules (S, Δ) are $\frac{\varepsilon}{\lambda_M(1+\mathbb{1}_{\text{REG}})}$ -uniformly-convergent w.r.t. $\|\cdot\|_M$ and the *additive error vector bound* $\text{AEV}(\mathbf{m}, \delta, \mathbf{x}, \mathbf{y}) \leftarrow \langle \text{AES}(\mathbf{m}_1, \frac{\delta}{g}, \mathbf{x}_1, \mathbf{y}_1), \dots, \text{AES}(\mathbf{m}_g, \frac{\delta}{g}, \mathbf{x}_g, \mathbf{y}_g) \rangle$. Now take μ to be the true objective value of \hat{h} and μ^* to be the true objective value of the optimal h^* (see thm. 4.5). Then, with probability at least $1 - \delta$, we have

- 1) $|\hat{\mu} - \mu| \leq \hat{\varepsilon} \leq \varepsilon$; &
- 2) $M^{\star\downarrow} \leq \mu^* \leq \mu \leq \hat{\mu} + \hat{\varepsilon} \leq M^{\star\downarrow} + 2\varepsilon$.

PROOF. Proof of thm. 4.6 is quite similar to that of thm. 4.5. The primary difference is that the union bound is now over g individual schedules and all timesteps, i.e., we now have the total failure probability of all possible tail bounds (taken on line 17) is no greater than

$$\sum_{i \in \mathcal{Z}} \sum_{t=1}^{\infty} \frac{\Delta_t}{g} = \frac{g}{g} \sum_{t=1}^{\infty} \Delta_t = \delta .$$

Note that while the *order* these bounds are taken is random, the braided structure takes no more than one bound for each (i, t_i) pair consisting of a *group* $i \in \mathcal{Z}$ and a *time index* $t_i \in \mathbb{Z}_+$, and said bounds *do not depend* on the order in which they are taken. As the bounds themselves are always taken at the same sample sizes, we need only sum over failure probabilities of all possible tail bounds that may be taken by the algorithm to union-bound the total failure probability. As in the proof of the linear algorithm, we conclude that all tail bounds that could possibly be taken by the algorithm (now over all groups $i \in \mathcal{Z}$ and all timesteps $t_i \in \mathbb{Z}_+$) hold simultaneously with probability at least $1 - \delta$.

We now condition on the event that all bounds taken are correct (which holds with probability at least $1 - \delta$). In this case, as in the linear algorithm, so long as the braided algorithm *terminates*, it produces a sufficiently accurate answer. The reasoning here is identical to that of thm. 4.5, as both algorithms share a termination condition.

Despite their similarity, it is not so straightforward as in the linear algorithm to show that the braided algorithm is *guaranteed to terminate* (i.e., it does not loop indefinitely). In particular, consider that while in alg. 1, the appropriate uniformly-convergent schedule is sufficient to guarantee termination eventually, this would not be so in alg. 2 *if it were to select groups arbitrarily* at each iteration, e.g., sampling from the same group at every iteration could loop indefinitely. However, this does not occur, due to the *group selection logic* on line 16. We consider first the 0-uniformly convergent schedule case, and then the general case, concluding eventual termination in both.

We first show that given a 0-uniformly convergent schedule, the algorithm will not loop indefinitely without sampling each group an infinite number of times, thus if the algorithm does not first terminate, any per-group time index vector \mathbf{t} will eventually be exceeded (componentwise). To conclude this, it is sufficient to show that for each group i , the algorithm will either iterate until it either terminates, or group i is selected. This is because group i will eventually have the optimal improvement:cost ratio, and thus be selected to sample on line 16, which we now show. Consider that, by the *strict monotonicity* assumption, there is always nonzero (positive) projected improvement gain to improving the error bound $\hat{\epsilon}_i$, and thus to selecting any group i to sample. However, selecting other groups *ad nauseam* will take their cost terms to ∞ , and thus their improvement:cost ratios to 0 (in the limit). It thus follows that, if the termination condition were never met, each group i would be sampled from an unbounded number of times, during which their error bound would converge as $\hat{\epsilon}_i \rightsquigarrow 0$, and thus $\hat{\epsilon} \rightsquigarrow \mathbf{0}$. Consequently, by the *continuity* assumption, it holds that $\hat{\mathbb{M}}^\uparrow \rightsquigarrow \mathbb{M}(h^*)$ and $\mathbb{M}^{*\downarrow} \rightsquigarrow \mathbb{M}(h^*)$, which implies termination on line 11.

The case of an $\frac{\epsilon}{\lambda_M(1+\mathbb{1}_{\text{REG}})}\|\cdot\|_M$ -uniformly-convergent schedule follows similarly, except now groups may cease sampling if their error bounds are nonzero, but sufficiently small so as to ϵ -estimate the objective. If this holds for all groups (which again happens eventually if termination does not occur first), we again meet the termination condition of line 11, hence we conclude guaranteed termination in this case.

With termination shown in all cases, we now conclude that alg. 2, produces a sufficiently accurate answer with the stated probability, by the same reasoning as in the analysis of alg. 1, i.e., thm. 4.5. \square

B A TRAVELER’S HANDBOOK TO PROGRESSIVE SAMPLING

In this appendix, we provide deeper intuition for our progressive sampling algorithms, with an emphasis on how they differ from standard progressive sampling approaches. We also describe how decisions are made as to which group to sample in the braided algorithm (alg. 2), i.e., under the conditional sampling model. Note that this material is provided purely to supplement understanding of the methods; all proofs related to these algorithms are detailed in appx. A.2.

B.1 Tricks of the Trade: the Magician’s Secrets, Revealed

Understanding progressive sampling algorithms essentially comes down to understanding *how union bounds are taken*, and *which tail bounds* may be taken and *when*, since correcting for the multiple comparisons problem is the central technical issue in such methods (e.g., an algorithm that computes bounds after every sample while using a union bound is extremely inefficient, because of the excessive union bound cost). For the most part, algs. 1 and 2 are standard progressive sampling methods, however there are some subtle details that obscure the simple reasoning at their core. In particular, under the mixture sampling model, sample sizes are not always known *a priori*, and in the conditional

sampling model, decisions are actually made dynamically that determine the order in which tail bounds are taken. Both of these decisions have ramifications that impact the core logic of a static alternating series of *sampling* and *bounding* steps, but as always, there is no real magic here: merely a few logical flourishes that permit these slight modifications to the standard flow of progressive sampling algorithms.

Under the joint sampling model, the linear progressive sampling algorithm is quite straightforward, as the total number of samples drawn b each timestep is completely known *a priori* (i.e., S_t from each group at timestep t), so a simple union bound over a deterministic schedule suffices. Under the mixture sampling model, there is some subtlety to why the algorithm works as it does, since the sample sizes at which tests are run are in fact a random variable dependent on the order in which groups are sampled. A naïve approach would be to consider a union bound over possible orders of sampling, or otherwise correct for the fact that multiple outcomes are possible, however this is ultimately not necessary, and such approaches would induce harmful corrective terms to probabilistic error-bounds and sample complexities. The simple trick we employ is to condition the algorithm on the infinite sequence of \mathcal{Z} samples drawn on line 10, and establish the schedule *after performing this conditioning operation*, thus samples drawn under the mixture sampling model are in fact samples from individual groups in a *known order*. Of course, we don't "really" know this order, nor do we perform this conditioning anywhere in the algorithm; this is an analytical technique, and the simple fact of its existence suffices to show correctness. This analysis does not actually require a randomized order of \mathcal{Z} samples, and in fact the algorithm works even if group identities are *adversarially selected*, as nowhere in the analysis do we actually assume sampled group identities (line 10) are *random*. However, it must be true under adversarial \mathcal{Z} selection that each $(\mathcal{X}, \mathcal{Y})$ pair drawn is *conditionally independently* given each group in the set \mathcal{Z} , thus the adversarial mixture sampling analysis is most useful in the context of mutually exclusive groups (i.e., singleton z samples).

Because of these design decisions, all three algorithms are able to use simple progressive sampling parameters, such as a single sampling schedule S and failure probability schedule Δ , albeit in slightly different ways. They also enjoy the characteristic sharpness of standard progressive sampling algorithms, with only an extra $\frac{1}{g}$ factor attached to failure probabilities (δ values) for the conditional sampling model, to accommodate the union bound over g simultaneous (braided) linear progressive sampling instances.

B.2 Selecting Where to Sample in the Braided Algorithm

We now discuss how the decision as to which group to sample from is made. Our reasoning here largely parallels that of property 3.8: we want to maximize the improvement made to the UCB-optimal \hat{h} . However, rather than apply the linear subderivative approximation, which is accurate for small changes (e.g., adding a single sample), we consider the impact of advancing the sampling schedule of each group by one timestep; for geometric schedules, this is a *multiplicative* — rather than an additive — change to the sample size. We must also include the cost of sampling when selecting where to sample, so this too enters the equation through the linear cost model $C_{1:g}$. Intuitively, the idea is essentially to select the group i for which the *ratio of projected improvement to cost* is maximized.

Unfortunately, there is a slight wrinkle in the algorithm: at each iteration, we can not simply greedily maximize the improvement:cost ratio from a single timestep of sampling group i , because this leads into a failure mode, wherein if a group's risk is near c , i.e., near maximal, it will never be sampled. There is no simple workaround, as the malfore function may not even be defined for inputs larger than c (for instance if $\mathcal{M}(s; \mathbf{w})$ is actually the function $c - W_0(c - s; \mathbf{w})$, i.e., geometric welfare through the reduction of the `LINEARPSUTILITY(...)` procedure of alg. 1). Our solution is to maximize not just the improvement:cost ratio of sampling for a *single* additional timestep, but over *any number* t of timesteps. For reasonable malfore functions and AEV bounds, when the empirical risk is unchanged after taking the

minimum with c (i.e., the expression $c \wedge \hat{\mathbb{E}}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}}[\ell \circ h]$ takes the value $\hat{\mathbb{E}}_{\mathbf{x}_{i,:}, \mathbf{y}_{i,:}}[\ell \circ h]$ on line 8 of alg. 2), we expect diminishing returns (in improvement:cost ratio) to further sampling, thus the $t = 1$ timestep greedy optimal choice should usually be selected.

In particular, given per-group uniform convergence bounds $\hat{\epsilon}_{1:g}$ and assuming each group is on timestep $t_{1:g}$ of their sampling schedule, we estimate the UCB improvement⁸ for sampling from group i for t timesteps as as

$$\tilde{\epsilon}_{j,t}^{(i)} \leftarrow \hat{\epsilon}_j \quad \text{for } i \neq j, \quad \& \quad \tilde{\epsilon}_{j,t}^{(i)} \leftarrow \hat{\epsilon}_j \sqrt{\frac{S_{t_j} \ln \frac{g}{\Delta_{t_j}}}{t + S_{t_j} \ln \frac{g}{\Delta_{t+t_j}}}} \quad \text{for } i = j \quad (22)$$

(see line 15). We then estimate the malfare improvement as $\Delta_{i,t} \leftarrow \hat{M}^\dagger - \mathcal{M}(j \mapsto c \wedge \hat{\mathbb{E}}_{\mathbf{x}_{j,:}, \mathbf{y}_{j,:}}[\ell \circ h] - \hat{s}_j^* + \tilde{\epsilon}_{j,t}^{(i)}; \mathbf{w})$, and compute the cost of sampling from group i for t timesteps as $C_i(S_{t+t_i} - S_{t_i})$. We then select the group with maximal projected improvement to sampling cost ratio, i.e., we select $i \leftarrow \operatorname{argmax}_{i \in \mathcal{Z}} \sup_{t \in \mathbb{Z}_+} \frac{\Delta_{i,t}}{C_i(S_{t+t_i} - S_{t_i})}$ (see line 16).

Note that even if the estimated improvement is not accurate, we still gain information; either the upper bound decreases more than expected, which decreases the relative value of further sampling from this group (in which case we are less likely to sample from this group in the future), or it decreased less than expected (likely due to selection bias), in which case we may decide to sample it again, but now more work is required to get the same reduction in confidence radius (increased cost), so sampling from another group may now be optimal.

Further Notes on Long-Term Planning. Optimizing over timestep count t , as well as group i , does seem like it may create some issues for the algorithm, however the impact is philosophically, computationally, and practically very small. In general, as long as some improvement to sampling a group is projected to happen *eventually*, evaluating the supremum over t exhaustively will eventually reach some t such that the cost is so large that no larger t can be optimal. For example, if the current (regret) malfare UCB is \hat{M}^\dagger , and improvement $\Delta_{i,t} > 0$ is projected after some timestep count t , then for cutoff timestep t^\dagger , defined as the smallest integer $t^\dagger > t$ such that

$$\frac{\hat{M}^\dagger}{C_i(S_{t^\dagger+t_i} - S_{t_i})} \leq \frac{\Delta_{i,t}}{C_i(S_{t+t_i} - S_{t_i})}, \quad \text{or equivalently, } S_{t^\dagger+t_i} \geq S_{t_i} + (S_{t+t_i} - S_{t_i}) \frac{\hat{M}^\dagger}{\Delta_{i,t}}, \quad (23)$$

the improvement:cost ratio is never maximized after more than t^\dagger additional timesteps. We thus conclude evaluating the supremum over t in group selection (line 16) is not computationally intractable.

Note also that even if sampling group i for t timesteps yields an optimal improvement:cost ratio at this iteration, the algorithm may change course on the next iteration. For example, if more improvement than projected occurs, then sampling from another group j , which now has a greater impact on the malfare (e.g., group i is no longer maximal, and thus inconsequential to egalitarian malfare) now optimizes the improvement:cost ratio, or if less improvement occurs than projected, then projected improvement of group i at the next timestep may decrease, after which another group may have the optimal projected improvement:cost ratio. In other words, despite some element of *long-term planning* with this supremum over t in group selection, the algorithm does not commit to sampling from a group for more than a single iteration, which is important, because long-term projections are likely to be inaccurate, as they are made with less information than the greedy-optimal group selection at each iteration.

⁸Note that we can't simply assume $\sqrt{\frac{1}{m}}$ rates of tail bound convergence, as δ may also be changing. Even incorporating the $\ln \frac{1}{\delta}$ terms characteristic of exponential tail bounds is not always entirely accurate; and when the schedule and bound are fully specified, in some cases it may be possible to produce a better estimate of the bound improvement. In particular, if $\text{AES}(\dots)$ is not data-dependent, we can simply take $\tilde{\epsilon}_{i,t}^{(i)} \leftarrow \text{AES}(S_{t+t_i}, \frac{\Delta_{t+t_i}}{g})$.